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Buku SEM Hibah LLDIKTI

by Anik Anekawati

SPATIAL AUTOREGRESSIVE MODEL DAN *SPATIAL ERROR MODEL* PADA *STRUCTURAL EQUATION MODELING*

Anik Anekawati
Bambang Widjanarko Otok
Purhadi
Sutikno
Syarifurrahman Hidayat
Mohammad Rofik

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KATA PENGANTAR

Puji syukur kehadirat Allah SWT yang telah melimpahkan rahmat, nikmat dan petunjuk-Nya sehingga buku ***Spatial Autoregressive Model*** **dan** ***Spatial Error Model*** **pada** ***Structural Equation Modeling*** ini telah dapat diselesaikan.

Spatial Autoregressive Model (SAR-SEM) dan *Spatial Error Model* pada *Structural Equation Modeling* (SERM-SEM) merupakan bagian dari pemodelan SEM spasial. Pemodelan ini dibutuhkan ketika terdapat variabel laten yang memiliki hubungan kausalitas sekaligus mempunyai pengaruh secara spasial. Untuk mengatasi permasalahan tersebut, unsur spasial perlu diikutsertakan ke dalam model SEM (SEM spasial). Pada saat analisis statistik yang melibatkan variabel laten dan sekaligus efek spasial, maka terdapat dua *framework* dalam pelibatan data spasial pada model SEM, yaitu pada tingkat model pengukuran atau pada model struktural. Bobot spasial yang menggambarkan *spatial spill-over effects* diletakkan pada model struktural lebih fleksibel dan informatif. Dengan menggantikan variabel laten pada model SEM dengan hasil estimasi nilai dari variabel itu sendiri, membuat model SEM spasial tidak menggunakan distribusi *error* model sama dengan model spasial tradisional. Oleh karena itu, buku ini akan memberikan perspektif yang berbeda dan relatif baru.

Penulis menyampaikan terima kasih kepada semua pihak yang ikut serta dalam penyusunan buku ini. Ucapan terima kasih juga disampaikan kepada Direktorat Jenderal Pendidikan Tinggi, Riset, dan Teknologi cq. Direktorat Riset, Teknologi, dan Pengabdian Kepada Masyarakat (DRTPM) atas bantuan pendanaan melalui skema hibah Penelitian Dasar Kompetitif Nasional (PDKN) dengan nomor kontrak 159/E5/P6.02.00.PT/2022; dan nomor kontrak turuan 159/E5/P6.02.00.PT/2022,001/SP2H/PEN-DRPM/SK-PDKN/ LPPM/UNIJA/V/2022.

Penulis menyadari masih banyak kekurangan sehingga membutuhkan masukan, kritik dan saran untuk penyempurnaan selanjutnya. Semua korespondensi dapat dilakukan dengan email anik@wiraraja.ac.id. Semoga buku ini dapat bermanfaat bagi pengembangan ilmu pengetahuan di Indonesia, khususnya bagi perkembangan *Spatial Structural Equation Modeling* (SEM Spasial).

September 2022

Tim Penulis

DAFTAR ISI

SPATIAL AUTOREGRESSIVE MODEL **DAN**
SPATIAL ERROR MODEL **PADA**
STRUCTURAL EQUATION MODELING

BAB 1 PENDAHULUAN

Penelitian sosial dan perilaku, sering kali melibatkan variabel laten. Salah satu teknik analisis statistik yang memiliki kemampuan untuk menganalisis pola hubungan antar variabel laten dan antara variabel laten dengan indikatornya adalah *Structural Equation Modeling* (SEM). SEM merupakan kombinasi teknik analisis data multivariat interdependensi dan dependensi, yaitu kombinasi analisis faktor konfirmatori dan analisis jalur. Schumacker dan Lomax (2004) menjelaskan bahwa SEM sering digambarkan dengan menggunakan model jalur dimana faktor-faktor dipandang sebagai variabel laten. Variabel yang dianalisis adalah variabel laten, yaitu variabel yang tidak dapat diobservasi secara langsung tetapi diukur melalui indikator-indikator terukur (variabel manifes).

Pada beberapa kasus, terdapat variabel laten yang memiliki hubungan kausalitas sekaligus mempunyai pengaruh secara spasial. Untuk mengatasi permasalahan tersebut, unsur spasial perlu diikutsertakan ke dalam model SEM (SEM spasial). Pada saat analisis statistik yang melibatkan variabel laten dan sekaligus efek spasial, maka terdapat dua *framework* dalam pelibatan data spasial pada model SEM, yaitu pada tingkat model pengukuran atau pada model struktural. Pada buku ini, bobot spasial yang menggambarkan *spatial spill-over effects* diletakkan pada model struktural. Pendekatan ini lebih fleksibel dan informatif daripada pemodelan dengan memberikan bobot spasial pada model pengukuran (Oud and Folmer, 2008).

Ruang lingkup buku ini adalah pemodelan *spatial structural equation modeling* (SEM spasial), khususnya untuk *spatial autoregressive model* pada *structural equation modeling* (SAR-SEM) dan *spatial error model* pada *structural equation modeling* (SERM-SEM). Pemodelan ini dimulai dengan mengestimasi variabel laten eksogen dan endogen untuk mendapatkan distrubusi dari *error* model tersebut menggunakan metode *weighted least square* (WLS). Oleh karena itu, distribusi *error* model SEM spasial tidak sama dengan distribusi *error* model spasial yang dikembangkan oleh Anselin (1988a).

Buku ini juga menguraikan statistik uji untuk uji dependensi spasial menggunakan uji *Langrange Multiplier* baik untuk model SAR-SEM maupun SERM-SEM. Selanjutnya, estimasi parameter model menggunakan metode *generalized spatial two-stage least squares* (GS2SLS) yang dikembangkan oleh Kelejian dan Prucha (1998, 1999). Model SAR-SEM menggunakan pendekatan *two stage least square* (2SLS)dan model SERM-SEM dengan pendekatan *Generalized Method of Moments* (GMM). Dan yang terakhir, uji parameter model SAR-SEM dan SERM-SEM secara simultan menggunakan *maximum likelihood ratio test* (MLRT). Buku ini juga dilengkapi lampiran yang berisikan bukti-bukti pendukung kajian matematis.

Untuk memperdalam pemahaman kajian matematika pada buku ini, pembaca dapat membaca contoh-contoh aplikasi kajian matematika ini pada artikel-artikel berikut ini: Anekawati dkk (2017; 2018; 2020a; 2020b).

BAB 2 MODEL SEM SPASIAL

2.1. Structural Equation Modeling (SEM)

Structural Equation Modeling (SEM) adalah teknik analisis statistika yang mengkombinasikan beberapa aspek yang terdapat pada *path analysis* dan analisis faktor konfirmatori untuk mengestimasi beberapa persamaan secara simultan. Terdapat dua karakteristik SEM yang membedakan dengan analisis regresi atau analisis multivariat lainnya. Perbedaan tersebut bahwa susunan *multiple interrelated dependence relationship* dispesifikasikan dalam bentuk model struktural dan diestimasi secara simultan dan mempunyai kemampuan untuk menunjukkan konstruk yang tidak teramati beserta hubungan yang ada di dalamnya serta melakukan perhitungan kesalahan pengukuran dalam proses estimasi.

Secara umum SEM mempunyai dua submodel, yaitu model pengukuran dan model struktural. Model struktural merupakan hubungan antara variabel laten independen (eksogen) dengan variabel laten dependen (endogen). Schumacker dan Lomax (2004) menuliskan model persamaan struktural adalah sebagai berikut:

$$\underset{q\times 1}{\boldsymbol{\eta}} = \underset{q\times q}{\mathbf{B}}\ \underset{q\times 1}{\boldsymbol{\eta}} + \underset{q\times p}{\boldsymbol{\Gamma}^*}\ \underset{p\times 1}{\boldsymbol{\xi}} + \underset{q\times 1}{\boldsymbol{\zeta}}\ . \tag{2.1}$$

Persamaan (2.1) dapat ditulis kembali sebagai berikut:

$$\underset{q\times 1}{\boldsymbol{\eta}} = \left(\underset{q\times q}{\mathrm{I}} - \underset{q\times q}{\mathbf{B}}\right)^{-1}\left(\underset{q\times p}{\boldsymbol{\Gamma}^*}\ \underset{p\times 1}{\boldsymbol{\xi}} + \underset{q\times 1}{\boldsymbol{\zeta}}\right), \tag{2.2}$$

dimana $\boldsymbol{\eta}$ adalah vektor variabel random endogen, $\boldsymbol{\xi}$ adalah vektor variabel random eksogen, $\mathbf{B}$ adalah matrik koefisien yang menunjukkan pengaruh hubungan variabel laten endogen terhadap variabel endogen lainnya dan $\boldsymbol{\Gamma}^*$ adalah matrik koefisien yang menunjukkan pengaruh hubungan variabel laten eksogen ($\boldsymbol{\xi}$)

terhadap variabel laten endogen ($\boldsymbol{\eta}$), sedangkan ζ adalah vektor random *error*, dengan nilai harapan sama dengan nol. Indeks q menunjukkan jumlah variabel laten endogen dan indeks p adalah jumlah variabel laten eksogen.

Asumsi yang harus dipenuhi dalam persamaan struktural (2.2) antara lain $E(\boldsymbol{\eta}) = \mathbf{0}$, $E(\xi) = \mathbf{0}$, $E(\zeta) = \mathbf{0}$, $E(\xi\zeta') = \mathbf{0}$, dimana ξ tidak berkorelasi dengan ζ dan **(I-B)** *nonsingular*.

Matrik kovarian dari variabel laten eksogen ξ adalah $\boldsymbol{\Phi}$, dimana $\boldsymbol{\Phi} = \begin{bmatrix} \sigma^2_{\xi_1} & \sigma^2_{\xi_1,\xi_2} & \cdots & \sigma^2_{\xi_1,\xi_p} \\ \sigma^2_{\xi_2,\xi_1} & \sigma^2_{\xi_2} & \cdots & \sigma^2_{\xi_2,\xi_p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^2_{\xi p,\xi_1} & \sigma^2_{\xi p,\xi_2} & \cdots & \sigma^2_{\xi_p} \end{bmatrix}$, sedangkan matrik kovarian dari kesalahan struktural $\boldsymbol{\zeta}$ adalah $\boldsymbol{\Psi}$, dimana $\Psi = \sigma^2_{\zeta_p}$.

Pada SEM, variabel laten dihubungkan dengan variabel teramati melalui model pengukuran. Tujuan model pengukuran ini adalah untuk mengukur dimensi-dimensi yang membentuk sebuah faktor. Model pengukuran merepresentasikan hubungan antara indikator-indikator dengan faktornya yang dievaluasi dengan menggunakan teknik analisis faktor konfirmatori atau *Confirmatori Factor Analysis* (CFA) (Kline, 2005).

Model pengukuran dapat dituliskan sebagai berikut (Schumacker dan Lomax, 2004):

$$\underset{Ax1}{\mathbf{x}} = \underset{Axp}{\boldsymbol{\Lambda}_\mathbf{x}} \underset{px1}{\boldsymbol{\xi}} + \underset{Ax1}{\boldsymbol{\delta}^*}, \text{ dimana } \boldsymbol{\delta}^* \sim N(\mathbf{0}, \boldsymbol{\Theta}_\delta) \quad (2.3)$$

$$\underset{B\times1}{\mathbf{y}} = \underset{B\times q}{\boldsymbol{\Lambda}_\mathbf{y}} \underset{q\times1}{\boldsymbol{\eta}} + \underset{B\times1}{\boldsymbol{\varepsilon}^*}, \text{ dimana } \boldsymbol{\varepsilon}^* \sim N(\mathbf{0}, \boldsymbol{\Theta}_{\varepsilon^*}) \quad (2.4)$$

Keterangan notasi dari persamaan (2.3) dan (2.4) adalah sebagi berikut: $\boldsymbol{\eta}$ adalah vektor variabel random endogen, ξ adalah vektor variabel random eksogen, $\mathbf{y}$ dan $\mathbf{x}$ merupakan vektor dari variabel yang diobservasi. $\boldsymbol{\Lambda}_\mathbf{y}$ dan $\boldsymbol{\Lambda}_\mathbf{x}$ merupakan matrik koefisien yang menunjukkan hubungan dari $\mathbf{y}$ ke $\boldsymbol{\eta}$ dan $\mathbf{x}$ ke ξ serta $\boldsymbol{\varepsilon}^*$ dan $\boldsymbol{\delta}^*$ adalah vektor *error* dari pengukuran $\mathbf{y}$ dan $\mathbf{x}$. Kesalahan pengukuran ini merupakan penambahan komponen yang mewakili ketidaksempurnaan inidikator-indikator dalam mengukur variabel laten terkait.

Indeks pada persamaan (2.3) dimana p menunjukkan jumlah variabel laten eksogen. Jumlah indikator dari variabel laten eksogen ke-i adalah a_i, dimana $i = 1,2,3,\dots,p$. Jumlah seluruh indikator pada variabel laten eksogen adalah $\sum_{i=1}^{p} a_i = A$, sehingga indeks A menunjukkan jumlah seluruh indikator pada variabel laten eksogen.

Secara umum matrik kovarian dari kesalahan pengukuran variabel teramati $\mathbf{x}$ dengan variabel laten eksogen sebanyak p dan jumlah seluruh indikator sebanyak A adalah

$$\underset{A\times A}{\mathbf{\Theta}_{\delta}} = diag\left(\sigma^2_{\delta_{(1)1}}, \sigma^2_{\delta_{(2)1}}, \cdots, \sigma^2_{\delta_{(a1)1}}, \sigma^2_{\delta_{(1)2}}, \sigma^2_{\delta_{(2)2}}, \cdots, \sigma^2_{\delta_{(a2)2}}, \cdots, \sigma^2_{\delta_{(1)p}}, \sigma^2_{\delta_{(2)p}}, \cdots, \sigma^2_{\delta_{(ap)p}}\right)$$

Indeks pada persamaan (2.4) dimana q menunjukkan jumlah variabel laten endogen. Jumlah indikator dari variabel laten endogen ke-j adalah b_j, dimana $j = 1,2,3,\dots,q$. Jumlah seluruh indikator pada variabel laten endogen adalah $\sum_{j=1}^{q} b_j = B$, sehingga indeks B menunjukkan jumlah seluruh indikator pada variabel laten endogen.

Secara umum matrik kovarian dari kesalahan pengukuran variabel teramati $\mathbf{y}$ dengan variabel laten endogen sebanyak satu dan jumlah seluruh indikator sebanyak B adalah $\underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}} = diag\left(\sigma^2_{\varepsilon_1^*}, \sigma^2_{\varepsilon_2^*}, \cdots, \sigma^2_{\varepsilon_B^*}\right)$.

2.2. Model Spasial

Anselin (1988a) mengembangkan model spasial dengan menggunakan data spasial *cross section*, yang secara umum dituliskan sebagai berikut:

$$\underset{T\times 1}{\mathbf{y}^*} = \underset{T\times(p+1)}{\mathbf{X}} \underset{(p+1)\times 1}{\boldsymbol{\beta}} + \lambda \underset{T\times T}{\mathbf{W}} \underset{T\times 1}{\mathbf{y}^*} + \underset{T\times 1}{\mathbf{u}}, \tag{2.5}$$

$$\underset{T\times 1}{\mathbf{u}} = \rho \underset{T\times T}{\mathbf{M}} \underset{T\times 1}{\mathbf{u}} + \underset{T\times 1}{\boldsymbol{\varepsilon}}, \tag{2.6}$$

dengan $\boldsymbol{\varepsilon} \sim N\left(\mathbf{0}, \sigma^2_{\varepsilon}\mathbf{I}\right)$ dan notasi-notasi tersebut adalah:

$\mathbf{y}^*$	:	vektor variabel endogen yang memiliki ketergantungan secara spasial
$\mathbf{X}$	:	matriks variabel eksogen
$\boldsymbol{\beta}$	:	vektor parameter model regresi
λ	:	koefisien spasial *autoregressive* yang nilainya $\lvert\lambda\rvert < 1$
ρ	:	koefisien spasial dalam *error* ε yang nilainya $\lvert\rho\rvert < 1$
$\mathbf{W},\mathbf{M}$	:	matriks penimbang spasial elemen diagonalnya bernilai 0
$\mathbf{u}$	:	vektor *error* regresi yang diasumsikan mempunyai efek spasial random dan juga *error* yang terautokorelasi secara spasial
$\boldsymbol{\varepsilon}$	:	vektor *error*
T	:	jumlah amatan atau lokasi (t=1, 2, 3, … , T)
p	:	jumlah variabel **X**

Dari persamaan (2.5), dapat diperoleh model turunannya, yaitu:

1. Jika $\rho = 0$ dan $\lambda = 0$ diperoleh model regresi linier OLS, yaitu regresi yang tidak mempunyai efek spasial dengan model sebagai berikut:

   $\mathbf{y}^* = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ dengan $\boldsymbol{\varepsilon} \sim N\left(\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}\right)$.

2. Jika $\rho = 0$ dan $\lambda \neq 0$ maka diperoleh model *Spatial Autoregressive Model* (SAR).

   Karena $\mathbf{u} = \rho\mathbf{M}\mathbf{u} + \boldsymbol{\varepsilon}$ dan $\rho = 0$ maka $\mathbf{u} = (0)\mathbf{M}\mathbf{u} + \boldsymbol{\varepsilon}$ didapatkan $\mathbf{u} = \boldsymbol{\varepsilon}$ sehingga persamaan model SAR adalah:

   $$\mathbf{y}^* = \mathbf{X}\boldsymbol{\beta} + \lambda\mathbf{W}\mathbf{y}^* + \boldsymbol{\varepsilon} \,. \qquad (2.7)$$

   Model ini megasumsikan bahwa proses *autoregressive* hanya pada variabel endogen.

3. Jika $\rho \neq 0$ dan $\lambda = 0$ diperoleh model *Spatial Error Model* (SERM).

   $\mathbf{y}^* = \mathbf{X}\boldsymbol{\beta} + \lambda\mathbf{W}\mathbf{y}^* + \mathbf{u}$

   $\mathbf{y}^* = \mathbf{X}\boldsymbol{\beta} + (0)\mathbf{W}\mathbf{y}^* + \mathbf{u}$

   $\mathbf{y}^* = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$

   karena $\mathbf{u} = \rho\mathbf{M}\mathbf{u} + \boldsymbol{\varepsilon}$

$$(\mathbf{I}-\rho\mathbf{M})\mathbf{u}=\boldsymbol{\varepsilon}$$

$$\mathbf{u}=(\mathbf{I}-\rho\mathbf{M})^{-1}\boldsymbol{\varepsilon}$$

Sehingga $\mathbf{y}^*=\mathbf{X}\boldsymbol{\beta}+\mathbf{u}$

$$\mathbf{y}^*=\mathbf{X}\boldsymbol{\beta}+(\mathbf{I}-\rho\mathbf{M})^{-1}\boldsymbol{\varepsilon}$$

$$(\mathbf{I}-\rho\mathbf{M})\mathbf{y}^*=(\mathbf{I}-\rho\mathbf{M})\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\varepsilon}$$

$$\mathbf{y}^*-\rho\mathbf{M}\mathbf{y}^*=\mathbf{X}\boldsymbol{\beta}-\rho\mathbf{M}\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\varepsilon}$$

$$\mathbf{y}^*=\rho\mathbf{M}\mathbf{y}^*+\mathbf{X}\boldsymbol{\beta}-\rho\mathbf{M}\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\varepsilon}\,. \qquad (2.8)$$

4. Jika $\rho\neq 0$ dan $\lambda\neq 0$ diperoleh model *Spatial Autoregressive Moving Average* (SARMA) dengan model sebagaimana persamaan (2.5).

2.3. Model SEM Spasial

Oud dan Folmer (2008) menuliskan model spasial *lag* (SAR) laten dengan meregresikan variabel laten yang diukur dari beberapa indikator dalam bentuk *Multiple Indicators Multiple Causes* (MIMIC) dimana variabel $\mathbf{y}$ (vektor dengan pengamatan pada variabel dependen) dimana digantikan dengan $\boldsymbol{\eta}$, variabel laten $\boldsymbol{\xi}=\mathbf{x}$ dan ketergantungan spasial pada variabel laten (model struktural) bukan pada variabel observasi/manifes. Persamaan model spasial lag (SAR) laten tersebut dituliskan sebagai berikut:

$$\boldsymbol{\eta}=\rho\mathbf{W}\boldsymbol{\eta}+\boldsymbol{\Gamma}^*\mathbf{x}+\boldsymbol{\zeta}\,. \qquad (2.9)$$

Demikian juga untuk model spasial *error* (SERM) laten, Oud dan Folmer (2008) menuliskan sebagai berikut:

$$\boldsymbol{\eta}=\boldsymbol{\Gamma}^*\mathbf{x}+\boldsymbol{\varepsilon}\,, \qquad (2.10)$$

dimana

$$\boldsymbol{\varepsilon}=(\mathbf{I}-\lambda\mathbf{M})^{-1}\boldsymbol{\zeta}\,. \qquad (2.11)$$

Persamaan (2.11) disubsitusikan ke persamaan (2.10) maka persamaan (2.10) dapat ditulis kembali sebagai berikut:

$$\boldsymbol{\eta}=\boldsymbol{\Gamma}^*\mathbf{x}+(\mathbf{I}-\lambda\mathbf{M})^{-1}\boldsymbol{\zeta}$$

$$(\mathbf{I}-\lambda\mathbf{M})\boldsymbol{\eta}=(\mathbf{I}-\lambda\mathbf{M})\boldsymbol{\Gamma}^*\mathbf{x}+\boldsymbol{\zeta}$$

$$\boldsymbol{\eta}-\lambda\mathbf{M}\boldsymbol{\eta}=\boldsymbol{\Gamma}^*\mathbf{x}-\lambda\mathbf{M}\boldsymbol{\Gamma}^*\mathbf{x}+\boldsymbol{\zeta}$$

$$\boldsymbol{\eta} = \lambda \mathbf{M}\boldsymbol{\eta} + \boldsymbol{\Gamma}^{*}\mathbf{x} - \lambda \mathbf{M}\boldsymbol{\Gamma}^{*}\mathbf{x} + \boldsymbol{\zeta}. \quad (2.12)$$

Pada model SEM terdapat variabel laten yang tidak dapat diukur secara langsung sebagai unit sampel. Oleh karena itu, untuk merepresentasi variabel laten dalam model SEM spasial digantikan oleh skor faktor sebagai suatu sampel unit yang terukur. Skor faktor didapatkan melalui estimasi variabel laten. Selanjutnya, istilah skor faktor endogen merupakan skor yang diperoleh dari hasil estimasi variabel laten endogen. Demkikan juga, skor faktor eksogen merupakan skor hasil estimasi vartiabel laten eksogen. Pada penelitian ini mengacu pada model SEM spasial dari Oud dan Folmer (2008) dan menetapkan beberapa hal, antara lain: (1). Skor faktor merupakan skor hasil estimasi variabel laten menggunakan metode PLS dan WLS; (2). tidak menggunakan model MIMIC (tidak ada variabel eksogen maupun endogen yang merupakan *observed variable*); (3). ketergantungan spasial pada variabel laten (model struktural) bukan pada variabel observasi; dan (4). Matrik $\mathbf{W}$ berukuran $T \times T$ menunjukkan ketergantungan spasial antar amatan atau lokasi. Model SEM spasial secara umum ditulis sebagai berikut (notasi disesuaikan dengan model spasial pada subbab di atas):

$$\underset{T\times 1}{\boldsymbol{l}} = \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}} + \lambda \underset{T\times T}{\mathbf{W}} \underset{T\times 1}{\boldsymbol{l}} + \underset{T\times 1}{\mathbf{u}}, \quad (2.13)$$

$$\underset{T\times 1}{\mathbf{u}} = \rho \underset{T\times T}{\mathbf{M}} \underset{T\times 1}{\mathbf{u}} + \underset{T\times 1}{\boldsymbol{\varepsilon}}, \quad (2.14)$$

atau

$$\underset{T\times 1}{\mathbf{u}} = \left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{M}}\right)^{-1} \underset{T\times 1}{\boldsymbol{\varepsilon}}, \quad (2.15)$$

dengan $\boldsymbol{\varepsilon} \sim N\left(\mathbf{0}, \sigma_{\varepsilon}^{2}\mathbf{I}\right)$ dan dimana

$\boldsymbol{l}$	:	vektor skor faktor endogen yang memiliki ketergantungan secara spasial
$\boldsymbol{K}$	:	matriks skor faktor eksogen
$\boldsymbol{\beta}$	:	vektor parameter model regresi
λ	:	koefisien spasial *autoregressive* yang nilainya $\lvert\lambda\rvert < 1$
ρ	:	koefisien spasial dalam *error* ε yang nilainya $\lvert\rho\rvert < 1$

$\mathbf{W},\mathbf{M}$	:	matriks pembobot spasial dengan elemen diagonalnya bernilai 0
$\mathbf{u}$	:	vektor *error* regresi yang diasumsikan mempunyai efek spasial random dan juga *error* yang terautokorelasi secara spasial
$\boldsymbol{\varepsilon}$	:	vektor *error* dengan $\boldsymbol{\varepsilon} \sim N\left(0, \sigma_\varepsilon^2 \mathbf{I}\right)$
T	:	jumlah amatan atau lokasi (t=1, 2, 3, … , T)
p	:	jumlah variabel $\boldsymbol{K}$

Model turunan dari persamaan (2.13), yaitu:

1. Jika $\rho = 0$ dan $\lambda \neq 0$ maka diperoleh model *Spatial Autoregressive Model* pada SEM (SAR-SEM).

   Karena $\mathbf{u} = \rho\mathbf{M}\mathbf{u} + \boldsymbol{\varepsilon}$ dan $\rho = 0$ maka $\mathbf{u} = (0)\mathbf{M}\mathbf{u} + \boldsymbol{\varepsilon}$ didapatkan $\mathbf{u} = \boldsymbol{\varepsilon}$ sehingga persamaan model SAR-SEM adalah:

   $$\underset{T\times1}{\boldsymbol{l}} = \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times1}{\boldsymbol{\beta}} + \lambda \underset{T\times T}{\mathbf{W}} \underset{T\times1}{\boldsymbol{l}} + \underset{T\times1}{\boldsymbol{\varepsilon}} \tag{2.16}$$

   Model ini mengasumsikan bahwa proses *autoregressive* hanya pada skor faktor endogen ($\boldsymbol{l}$).

2. Jika $\rho \neq 0$ dan $\lambda = 0$ diperoleh model *Spatial Error Model* pada SEM (SERM-SEM).

   $$\underset{T\times1}{\boldsymbol{l}} = \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times1}{\boldsymbol{\beta}} + \lambda \underset{T\times T}{\mathbf{W}} \underset{T\times1}{\boldsymbol{l}} + \underset{T\times1}{\mathbf{u}}$$

   $$\underset{T\times1}{\boldsymbol{l}} = \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times1}{\boldsymbol{\beta}} + (0) \underset{T\times T}{\mathbf{W}} \underset{T\times1}{\boldsymbol{l}} + \underset{T\times1}{\mathbf{u}}$$

   sehingga model SERM pada SEM adalah:

   $$\underset{T\times1}{\boldsymbol{l}} = \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times1}{\boldsymbol{\beta}} + \underset{T\times1}{\mathbf{u}} \tag{2.17}$$

karena $\underset{T\times1}{\mathbf{u}} = \rho \underset{T\times T}{\mathbf{M}} \underset{T\times1}{\mathbf{u}} + \underset{T\times1}{\boldsymbol{\varepsilon}}$, $\left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{M}}\right) \underset{T\times1}{\mathbf{u}} = \underset{T\times1}{\boldsymbol{\varepsilon}}$, dan $\underset{T\times1}{\mathbf{u}} = \left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{M}}\right)^{-1} \underset{T\times1}{\boldsymbol{\varepsilon}}$, maka model SERM-SEM dapat ditulis kembali $\underset{T\times1}{\boldsymbol{l}} = \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times1}{\boldsymbol{\beta}} + \left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{M}}\right)^{-1} \underset{T\times1}{\boldsymbol{\varepsilon}}$ atau $\left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{M}}\right) \underset{T\times1}{\boldsymbol{l}} = \left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{M}}\right) \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times1}{\boldsymbol{\beta}} + \underset{T\times1}{\boldsymbol{\varepsilon}}$.

Model SERM-SEM secara umum dapat juga ditulis sebagai berikut:

$$\underset{T\times 1}{\boldsymbol{l}} = \rho \underset{T\times T}{\mathbf{M}} \underset{T\times 1}{\boldsymbol{l}} + \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}} - \rho \underset{T\times T}{\mathbf{M}} \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}} + \underset{T\times 1}{\boldsymbol{\varepsilon}} . \tag{2.18}$$

BAB 3 | ESTIMASI VARIABEL LATEN DAN DISTRIBUSI SKOR FAKTOR

Pada saat memodelkan SEM spasial dibutuhkan nilai skor faktor. Skor faktor merupakan hasil estimasi dari masing-masing variabel laten, baik endogen maupun eksogen pada model pengukuran. Metode yang digunakan adalah *weighted least square* (WLS), yaitu dengan cara meminimumkan jumlah kuadrat *error* yang diboboti matrik varian *error*.

Asumsi-asumsi untuk proses estimasi variabel laten adalah sebagai berikut:

1. Nilai *loading* faktor $\mathbf{\Lambda_x}$, $\mathbf{\Lambda_y}$, matrik varian *error* $\mathbf{\Theta}_{\delta}$, dan $\mathbf{\Theta}_{\varepsilon^*}$ adalah konstan.
2. Matrik bobot spasial $\mathbf{W}_T = \mathbf{M}_T$.
3. Semua elemen diagonal matrik bobot spasial $\mathbf{W}_T$ adalah nol.
4. Matrik $\left(\mathbf{I} - \lambda \mathbf{W}_T\right)$ adalah nonsingular dengan $|\lambda| < 1$.
5. ε_i berdistribusi identik (bersifat independen).
6. Bersifat linier dalam variabel.

3.1. Estimasi Variabel Laten Eksogen

Persamaan model pengukuran variabel laten eksogen sebagaimana persamaan (2.3) yaitu $\underset{Ax1}{\mathbf{x}} = \underset{Axp}{\mathbf{\Lambda_x}} \underset{px1}{\boldsymbol{\xi}} + \underset{Ax1}{\boldsymbol{\delta}^*}$. Persamaan model pengukuran dari variabel laten eksogen dapat ditulis kembali dalam bentuk matrik sebagai berikut:

$$\underset{A\times 1}{\begin{bmatrix} x_{(1)1} \\ x_{(2)1} \\ \vdots \\ x_{(a_1)1} \\ x_{(1)2} \\ x_{(2)2} \\ \vdots \\ x_{(a_2)2} \\ \vdots \\ x_{(1)p} \\ x_{(2)p} \\ \vdots \\ x_{(a_p)p} \end{bmatrix}} = \underset{A\times p}{\begin{bmatrix} \lambda_{(1)1} & 0 & \cdots & 0 \\ \lambda_{(2)1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{(a_1)1} & 0 & \cdots & 0 \\ 0 & \lambda_{(1)2} & \cdots & 0 \\ 0 & \lambda_{(2)2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda_{(a_2)2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{(1)p} \\ 0 & 0 & \cdots & \lambda_{(2)p} \\ \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{(a_p)p} \end{bmatrix}} \underset{p\times 1}{\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_p \end{bmatrix}} + \underset{A\times 1}{\begin{bmatrix} \delta_{(1)1} \\ \delta_{(2)1} \\ \vdots \\ \delta_{(a_1)1} \\ \delta_{(1)2} \\ \delta_{(2)2} \\ \vdots \\ \delta_{(a_2)2} \\ \vdots \\ \delta_{(1)p} \\ \delta_{(2)p} \\ \vdots \\ \delta_{(a_p)p} \end{bmatrix}}$$

Distribusi vektor variabel teramati $\mathbf{x}$ sebagaimana persamaan (2.3) dimana $\boldsymbol{\delta}^* \sim N\left(\mathbf{0},\boldsymbol{\Theta}_\delta\right)$ akan diurai pada Lemma 1.

Lemma 1

Jika $\mathbf{x} = \boldsymbol{\Lambda}_{\mathbf{x}}\boldsymbol{\xi} + \boldsymbol{\delta}^*$ dimana $\boldsymbol{\delta}^* \sim N\left(\mathbf{0},\boldsymbol{\Theta}_\delta\right)$ dengan asumsi $\boldsymbol{\Lambda}_x$ dan $\boldsymbol{\Theta}_\delta$ konstan, maka distribusi $\mathbf{x} \sim N\left(\boldsymbol{\Lambda}_{\mathbf{x}}\boldsymbol{\xi},\boldsymbol{\Theta}_\delta\right)$.

Bukti Lemma 1

Distribusi $\mathbf{x}$ didapatkan melalui sifat-sifat nilai ekpektasi dan variansi dari suatu variabel random yang didasarkan pada Teorema A.1 dan Teorema A.3 (Lampiran 1), yaitu:

$E\left(\mathbf{x}\right) = E\left(\boldsymbol{\Lambda}_{\mathbf{x}}\boldsymbol{\xi} + \boldsymbol{\delta}^*\right) = \boldsymbol{\Lambda}_x\boldsymbol{\xi}$ dan

$\mathrm{var}\left(\mathbf{x}\right) = \mathrm{var}\left(\boldsymbol{\Lambda}_{\mathbf{x}}\boldsymbol{\xi} + \boldsymbol{\delta}^*\right) = \mathrm{var}\left(\boldsymbol{\Lambda}_{\mathbf{x}}\boldsymbol{\xi}\right) + \mathrm{var}\left(\boldsymbol{\delta}^*\right) = \mathbf{0} + \boldsymbol{\Theta}_\delta = \boldsymbol{\Theta}_\delta$, sehingga distribusi $\mathbf{x}$

adalah $\underset{A\times 1}{\mathbf{x}} \sim N\left(\underset{A\times p}{\boldsymbol{\Lambda}_{\mathbf{x}}}\,\underset{p\times 1}{\boldsymbol{\xi}},\underset{A\times A}{\boldsymbol{\Theta}_\delta}\right)$.

Estimasi skor faktor eksogen menggunakan metode WLS dan hasilnya dipaparkan melalui Teorema 1.

Teorema 1.

Jika $\mathbf{x} = \mathbf{\Lambda}_x \boldsymbol{\xi} + \boldsymbol{\delta}^*$ dimana $\boldsymbol{\delta}^* \sim N(\mathbf{0}, \mathbf{\Theta}_\delta)$ dengan asumsi $\mathbf{\Lambda}_x$ dan $\mathbf{\Theta}_\delta$ konstan, maka hasil estimasi ξ dengan menggunakan metode WLS adalah $\sum_{t=1}^{T} \xi_t = \left(\mathbf{\Lambda}_x' \mathbf{\Theta}_\delta^{-1} \mathbf{\Lambda}_x\right)^{-1} \left(\mathbf{\Lambda}_x' \mathbf{\Theta}_\delta^{-1}\right) \sum_{t=1}^{T} \mathbf{x}_t$ dimana matrik $\left(\mathbf{\Lambda}_x' \mathbf{\Theta}_\delta^{-1} \mathbf{\Lambda}_x\right)$ adalah matrik diagonal dengan elemen selain diagonal utama adalah nol.

Bukti Teorema 1

Diketahui $\mathbf{x} = \mathbf{\Lambda}_x \boldsymbol{\xi} + \boldsymbol{\delta}^*$ dimana $\boldsymbol{\delta}^* \sim N(\mathbf{0}, \mathbf{\Theta}_\delta)$ dengan asumsi $\mathbf{\Lambda}_x$ dan $\mathbf{\Theta}_\delta$ konstan. Lemma 1 telah membuktikan bahwa distribusi $\mathbf{x}$ adalah $\mathbf{x} \sim N(\mathbf{\Lambda}_x \boldsymbol{\xi}, \mathbf{\Theta}_\delta)$. Misalkan diberikan T sampel random dari variabel random $\mathbf{x}$

$$\left(x_{(1)11}, x_{(2)12}, \ldots, x_{(a_1)1T}, x_{(1)21}, x_{(2)22}, \ldots, x_{(a_2)2T}, \ldots, x_{(1)P1}, x_{(2)pT}, \ldots,\right.$$

$$\left., x_{(a_p)pT}, \xi_{11}, \xi_{12}, \ldots, \xi_{1T}, \xi_{21}, \xi_{22}, \ldots, \xi_{2T}, \ldots, \xi_{p1}, \xi_{p2}, \ldots, \xi_{pT}\right),$$

dengan $t = 1,2,3,\ldots,T$ maka $\underset{A\times 1}{\mathbf{x}_t} \sim N\left(\underset{A\times p}{\mathbf{\Lambda}_x} \underset{p\times 1}{\xi_t}, \underset{A\times A}{\mathbf{\Theta}_\delta}\right)$. Fungsi probabilitas dari $\mathbf{x}_t$ adalah $f(\mathbf{x}_t) = (2\pi)^{-\frac{A}{2}} |\mathbf{\Theta}_\delta|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\mathbf{x}_t - \mathbf{\Lambda}_x \xi_t) \mathbf{\Theta}_\delta^{-1} (\mathbf{x}_t - \mathbf{\Lambda}_x \xi_t)\right\}$.

Fungsi *likelihood* adalah , $L(\xi, \mathbf{\Theta}_\delta) = \prod_{t=1}^{T} f(\mathbf{x}_t)$ dengan menggantikan nilai $\mathbf{x}_t$ maka fungsi *likelihood* menjadi

$$L(\xi, \mathbf{\Theta}_\delta) = (2\pi)^{-\frac{AT}{2}} |\mathbf{\Theta}_\delta|^{-\frac{T}{2}} \exp\left\{-\frac{1}{2} \sum_{t=1}^{T} (\mathbf{x}_t - \mathbf{\Lambda}_x \xi_t) \mathbf{\Theta}_\delta^{-1} (\mathbf{x}_t - \mathbf{\Lambda}_x \xi_t)\right\}.$$

Jika disederhanakan maka fungsi *likelihood* dapat dituliskan kembali

$$L(\xi, \mathbf{\Theta}_\delta) = (2\pi)^{-\frac{AT}{2}} |\mathbf{\Theta}_\delta|^{-\frac{T}{2}} \exp\left\{-\frac{1}{2} Q_x\right\}$$

dengan $\underset{1\times 1}{Q_x} = \sum_{t=1}^{T} \left(\underset{A\times 1}{\mathbf{x}_t} - \underset{A\times p}{\mathbf{\Lambda}_x} \underset{p\times 1}{\xi_t}\right) \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \left(\underset{A\times 1}{\mathbf{x}_t} - \underset{A\times p}{\mathbf{\Lambda}_x} \underset{p\times 1}{\xi_t}\right)$.

Variabel laten $\boldsymbol{\xi}$ diestimasi menggunakan metode WLS dengan optimasi $L(\boldsymbol{\xi},\boldsymbol{\Theta}_\delta)$. Berikut akan diurai optimasi $L(\boldsymbol{\xi},\boldsymbol{\Theta}_\delta)$ dengan memaksimumkan $L(\boldsymbol{\xi},\boldsymbol{\Theta}_\delta) \Leftrightarrow \min Q_x$.

$$Q_x = \sum_{t=1}^{T} (\mathbf{x}_t - \boldsymbol{\Lambda}_x \boldsymbol{\xi}_t) \boldsymbol{\Theta}_\delta^{-1} (\mathbf{x}_t - \boldsymbol{\Lambda}_x \boldsymbol{\xi}_t)$$

$$Q_x = \sum_{t=1}^{T} \underset{1\times A}{\mathbf{x}_t} \underset{A\times A}{\boldsymbol{\Theta}_\delta^{-1}} \underset{A\times 1}{\mathbf{x}_t} - 2\sum_{t=1}^{T} \underset{1\times p}{\boldsymbol{\xi}_t^{'}} \underset{p\times A}{\boldsymbol{\Lambda}_x^{'}} \underset{A\times A}{\boldsymbol{\Theta}_\delta^{-1}} \underset{A\times 1}{\mathbf{x}_t} + \sum_{t=1}^{T} \underset{1\times p}{\boldsymbol{\xi}_t^{'}} \underset{p\times A}{\boldsymbol{\Lambda}_x^{'}} \underset{A\times A}{\boldsymbol{\Theta}_\delta^{-1}} \underset{A\times p}{\boldsymbol{\Lambda}_x} \underset{p\times 1}{\boldsymbol{\xi}_t}$$

Selanjutnya dicari turunan parsial dari Q_x terhadap $\boldsymbol{\xi}$ adalah:

$$\frac{\partial Q_x}{\partial \boldsymbol{\xi}} = -2\sum_{t=1}^{T} \underset{p\times A}{\boldsymbol{\Lambda}_x^{'}} \underset{A\times A}{\boldsymbol{\Theta}_\delta^{-1}} \underset{A\times 1}{\mathbf{x}_t} + 2\sum_{t=1}^{T} \underset{p\times A}{\boldsymbol{\Lambda}_x^{'}} \underset{A\times A}{\boldsymbol{\Theta}_\delta^{-1}} \underset{A\times p}{\boldsymbol{\Lambda}_x} \underset{p\times 1}{\boldsymbol{\xi}_t}.$$

Jika hasil turunan parsial dari Q_x terhadap ξ disamakan dengan nol, maka didapatkan nilai skor faktor dari ξ, yaitu:

$$\sum_{t=1}^{T} \underset{p\times A}{\boldsymbol{\Lambda}_x^{'}} \underset{A\times A}{\boldsymbol{\Theta}_\delta^{-1}} \underset{A\times p}{\boldsymbol{\Lambda}_x} \underset{p\times 1}{\boldsymbol{\xi}_t} = \sum_{t=1}^{T} \underset{p\times A}{\boldsymbol{\Lambda}_x^{'}} \underset{A\times A}{\boldsymbol{\Theta}_\delta^{-1}} \underset{A\times 1}{\mathbf{x}_t}$$

$$\left(\underset{p\times A}{\boldsymbol{\Lambda}_\mathbf{x}^{'}} \underset{A\times A}{\boldsymbol{\Theta}_\delta^{-1}} \underset{A\times p}{\boldsymbol{\Lambda}_\mathbf{x}} \right) \sum_{t=1}^{T} \underset{p\times 1}{\hat{\boldsymbol{\xi}}_t} = \left(\underset{p\times A}{\boldsymbol{\Lambda}_\mathbf{x}^{'}} \underset{A\times A}{\boldsymbol{\Theta}_\delta^{-1}} \right) \sum_{t=1}^{T} \underset{A\times 1}{\mathbf{x}_t}$$

$$\sum_{t=1}^{T} \underset{p\times 1}{\hat{\boldsymbol{\xi}}_t} = \left(\underset{p\times A}{\boldsymbol{\Lambda}_\mathbf{x}^{'}} \underset{A\times A}{\boldsymbol{\Theta}_\delta^{-1}} \underset{A\times p}{\boldsymbol{\Lambda}_\mathbf{x}} \right)^{-1} \left(\underset{p\times A}{\boldsymbol{\Lambda}_\mathbf{x}^{'}} \underset{A\times A}{\boldsymbol{\Theta}_\delta^{-1}} \right) \sum_{t=1}^{T} \underset{A\times 1}{\mathbf{x}_t}, \qquad (3.1)$$

dimana matrik $\left(\underset{p\times A}{\boldsymbol{\Lambda}_\mathbf{x}^{'}} \underset{A\times A}{\boldsymbol{\Theta}_\delta^{-1}} \underset{A\times p}{\boldsymbol{\Lambda}_\mathbf{x}} \right)$ adalah matrik diagonal dengan elemen selain diagonal utama adalah nol (bukti pada Lampiran 2Lampiran).

Distribusi matrik random pengamatan dari vektor variabel teramati $\mathbf{x}$ sebagaimana persamaan (2.3) dimana $\boldsymbol{\delta}^* \sim N(\mathbf{0},\boldsymbol{\Theta}_\delta)$ dengan asumsi bahwa nilai *loading* faktor $\boldsymbol{\Lambda}_\mathbf{x}$ dan matrik varian *error* $\boldsymbol{\Theta}_\delta$ adalah konstan akan diurai pada Teorema 2.

Teorema 2.

Jika $\mathbf{x} = \boldsymbol{\Lambda}_x \boldsymbol{\xi} + \boldsymbol{\delta}^*$ dimana $\boldsymbol{\delta}^* \sim N(\mathbf{0},\boldsymbol{\Theta}_\delta)$ dengan asumsi $\boldsymbol{\Lambda}_\mathbf{x}$ dan $\boldsymbol{\Theta}_\delta$ konstan, maka distribusi matrik random pengamatan dari vektor variabel teramati $\mathbf{x}$ adalah $\mathbf{X} \sim N_{A,T}\left(\mathbf{\Lambda_x \xi_t e}', \mathbf{\Theta}_\delta \otimes \mathbf{I}_T\right)$ dimana $\mathbf{e}_{T\times 1} = (1,\ldots,1)'$.

Bukti Teorema 2.

Persamaan (3.1) dituliskan kembali dalam bentuk matrik adalah sebagai berikut:

$$\underset{p\times 1}{\begin{pmatrix} \xi_{11}+\xi_{12}+\ldots+\xi_{1T} \\ \xi_{21}+\xi_{22}+\ldots+\xi_{2T} \\ \vdots \\ \xi_{p1}+\xi_{p2}+\ldots+\xi_{pT} \end{pmatrix}} = \left(\underset{p\times A}{\mathbf{\Lambda}_x'}\,\underset{A\times A}{\mathbf{\Theta}_\delta'}\,\underset{A\times p}{\mathbf{\Lambda}_x}\right)^{-1}\left(\underset{p\times A}{\mathbf{\Lambda}_x'}\,\underset{A\times A}{\mathbf{\Theta}_\delta^{-1}}\right)\underset{A\times 1}{\begin{bmatrix} x_{(1)11}+x_{(1)12}+\ldots+x_{(1)1T} \\ x_{(2)11}+x_{(2)12}+\ldots+x_{(2)1T} \\ \vdots \\ x_{(a_1)11}+x_{(a_1)12}+\ldots+x_{(a_1)1T} \\ x_{(1)21}+x_{(1)22}+\ldots+x_{(1)2T} \\ x_{(2)21}+x_{(2)22}+\ldots+x_{(2)2T} \\ \vdots \\ x_{(a_2)21}+x_{(a_2)22}+\ldots+x_{(a_2)2T} \\ \vdots \\ x_{(1)p1}+x_{(1)p2}+\ldots+x_{(1)pT} \\ x_{(2)p1}+x_{(2)p2}+\ldots+x_{(2)pT} \\ \vdots \\ x_{(a_p)p1}+x_{(a_p)p2}+\ldots+x_{(a_p)pT} \end{bmatrix}} \qquad (3.2)$$

Jika persamaan (3.2) dibuat persamaan dalam bentuk matrik yang memuat masing-masing elemen x dan $\hat{\xi}$ adalah

$$\underset{p\times T}{\begin{pmatrix} \hat{\xi}_{11} & \hat{\xi}_{12} & \cdots & \hat{\xi}_{1T} \\ \hat{\xi}_{21} & \hat{\xi}_{22} & \cdots & \hat{\xi}_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\xi}_{p1} & \hat{\xi}_{p2} & \cdots & \hat{\xi}_{pT} \end{pmatrix}} = \left(\underset{p\times A}{\mathbf{\Lambda}_x'}\,\underset{A\times A}{\mathbf{\Theta}_\delta^{-1}}\,\underset{p\times A}{\mathbf{\Lambda}_x}\right)^{-1}\left(\underset{p\times A}{\mathbf{\Lambda}_x'}\,\underset{A\times A}{\mathbf{\Theta}_\delta^{-1}}\right)\underset{A\times T}{\begin{bmatrix} x_{(1)11} & x_{(1)12} & \cdots & x_{(1)1T} \\ x_{(2)11} & x_{(2)12} & \cdots & x_{(2)1T} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(a_1)11} & x_{(a_1)12} & \cdots & x_{(a_1)1T} \\ x_{(1)21} & x_{(1)22} & \cdots & x_{(1)2T} \\ x_{(2)21} & x_{(2)22} & \cdots & x_{(2)2T} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(a_2)21} & x_{(a_2)22} & \cdots & x_{(a_2)2T} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(1)p1} & x_{(1)p2} & \cdots & x_{(1)pT} \\ x_{(2)p1} & x_{(2)p2} & \cdots & x_{(2)pT} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(a_p)p1} & x_{(a_p)p2} & \cdots & x_{(a_p)pT} \end{bmatrix}} \qquad (3.3)$$

Pada persamaan (3.3) akan dimisalkan

$$\begin{bmatrix} x_{(1)11} & x_{(1)12} & \cdots & x_{(1)1T} \\ x_{(2)11} & x_{(2)12} & \cdots & x_{(2)1T} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(a_1)11} & x_{(a_1)12} & \cdots & x_{(a_1)1T} \\ x_{(1)21} & x_{(1)22} & \cdots & x_{(1)2T} \\ x_{(2)21} & x_{(2)22} & \cdots & x_{(2)2T} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(a_2)21} & x_{(a_2)22} & \cdots & x_{(a_2)2T} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(1)p1} & x_{(1)p2} & \cdots & x_{(1)pT} \\ x_{(2)p1} & x_{(2)p2} & \cdots & x_{(2)pT} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(a_p)p1} & x_{(a_p)p2} & \cdots & x_{(a_p)pT} \end{bmatrix}_{A\times T} = \mathbf{X}$$

dimana $\mathbf{X}$ adalah matrik random pengamatan dari vektor variabel teramati $\mathbf{x}$ dengan ukuran $A\times T$ Berdasarkan Definisi B.1 (Lampiran 1) maka distribusi matrik $\mathbf{X}'$ adalah $\mathbf{X} \sim N_{T,A}\left(\underset{T\times 1}{\mathbf{e}'}\underset{1\times p}{\xi_t}\underset{p\times A}{\mathbf{\Lambda}_\mathbf{x}'}, \underset{T\times T}{\mathbf{I}_T} \otimes \underset{A\times A}{\mathbf{\Theta}_\delta}\right)$ dimana $\mathbf{e}_{T\times 1} = (1,\ldots,1)$. Berdasarkan Teorema B.1 (Lampiran 1) maka distribusi $\mathbf{X}$ adalah $\mathbf{X} \sim N_{A,T}\left(\underset{A\times p}{\mathbf{\Lambda}_x}\underset{p\times 1}{\xi_t}\underset{1\times T}{\mathbf{e}'}, \underset{A\times A}{\mathbf{\Theta}_\delta} \otimes \underset{T\times T}{\mathbf{I}_T}\right)$.

Pada persamaan (3.3) dengan memisalkan matrik $\begin{pmatrix} \hat{\xi}_{11} & \hat{\xi}_{12} & \cdots & \hat{\xi}_{12} \\ \hat{\xi}_{21} & \hat{\xi}_{22} & \cdots & \hat{\xi}_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\xi}_{p1} & \hat{\xi}_{p2} & \cdots & \hat{\xi}_{pT} \end{pmatrix}_{p\times T} = \boldsymbol{K}'$ maka persamaan persamaan (3.3) dapat disederhanakan menjadi

$$\underset{p\times T}{\boldsymbol{K}'} = \left(\underset{p\times A}{\mathbf{\Lambda}_\mathbf{x}'}\underset{A\times A}{\mathbf{\Theta}_\delta^{-1}}\underset{A\times p}{\mathbf{\Lambda}_\mathbf{x}}\right)^{-1}\left(\underset{p\times A}{\mathbf{\Lambda}_\mathbf{x}'}\underset{A\times A}{\mathbf{\Theta}_\delta^{-1}}\right)\underset{A\times T}{\mathbf{X}}. \tag{3.4}$$

Distribusi K akan dipaparkan pada Teorema 3 berikut.

Teorema 3.

Jika distribusi matrik random pengamatan dari vektor variabel teramati $\mathbf{x}$ adalah $\mathbf{X} \sim N_{A,T}\left(\mathbf{\Lambda_x \xi}_t \mathbf{e}', \mathbf{\Theta}_\delta \otimes \mathbf{I}_T\right)$ dan $\boldsymbol{K}' = \left(\mathbf{\Lambda}_\mathbf{x}' \mathbf{\Theta}_\delta^{-1} \mathbf{\Lambda}_\mathbf{x}\right)^{-1}\left(\mathbf{\Lambda}_\mathbf{x}' \mathbf{\Theta}_\delta^{-1}\right)\mathbf{X}$ maka distribusi dari $\boldsymbol{K}$ adalah $\underset{T\times p}{\boldsymbol{K}} \sim N_{T,p}\left(\mathbf{e}\mathbf{\xi}_t', \mathbf{I}_T \otimes \left(\mathbf{\Lambda}_\mathbf{x}' \mathbf{\Theta}_\delta^{-1} \mathbf{\Lambda}_\mathbf{x}\right)^{-1}\right)$.

Bukti Teorema 3.

Definisi fungsi karakteristik dari matrik random X adalah $\phi_\mathbf{X}(\mathbf{Z}) = E\left[etr\left(\iota \mathbf{X}\mathbf{Z}'\right)\right]$ dengan $\iota = \sqrt{-1}$. Jika bagian dari persamaan (3.4) dimisalkan $\left(\underset{p\times A}{\mathbf{\Lambda}_\mathbf{x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \underset{A\times p}{\mathbf{\Lambda}_\mathbf{x}}\right)^{-1}\left(\underset{p\times A}{\mathbf{\Lambda}_\mathbf{x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}}\right) = \underset{A\times T}{\mathbf{P}}$ maka persamaan (3.4) dapat disederhanakan menjadi $\underset{p\times T}{\boldsymbol{K}'} = \underset{p\times A}{\mathbf{P}} \underset{A\times T}{\mathbf{X}}$. Fungsi karakteristik dari $\boldsymbol{K}'$ dituliskan sebagai berikut:

$$\phi_{\mathbf{K}'}(\mathbf{Z}) = E\left[etr\left(\iota \underset{p\times T}{\boldsymbol{K}'} \underset{T\times p}{\mathbf{Z}'}\right)\right] = E\left[etr\left(\iota\left(\underset{p\times A}{\mathbf{P}} \underset{A\times T}{\mathbf{X}}\right)\underset{T\times p}{\mathbf{Z}'}\right)\right] = E\left[etr\left(\iota \underset{p\times A}{\mathbf{P}}\left(\underset{A\times T}{\mathbf{X}} \underset{T\times p}{\mathbf{Z}'}\right)\right)\right].$$

Berdasarkan Teorema C.1 (Lampiran 1) maka fungsi karakteristik dari $\boldsymbol{K}'$ dapat diubah menjadi $\phi_{\mathbf{K}'}(\mathbf{Z}) = E\left[etr\left(\iota\left(\underset{A\times T}{\mathbf{X}} \underset{T\times p}{\mathbf{Z}'}\right)\underset{p\times A}{\mathbf{P}}\right)\right] = E\left[etr\left(\iota \underset{A\times T}{\mathbf{X}}\left(\underset{T\times p}{\mathbf{Z}'} \underset{p\times A}{\mathbf{P}}\right)\right)\right]$. Misalkan $\underset{T\times A}{\mathbf{Z}_1'} = \underset{T\times p}{\mathbf{Z}'} \underset{p\times A}{\mathbf{P}}$ maka fungsi karakteristik dari $\boldsymbol{K}'$ dapat diubah kembali menjadi $\phi_{\mathbf{K}'}(\mathbf{Z}) = E\left[etr\left(\iota \underset{A\times T}{\mathbf{X}} \underset{T\times A}{\mathbf{Z}'}\right)\right]$. Distribusi $\mathbf{X}$ adalah $\mathbf{X} \sim N_{A,T}\left(\underset{A\times p}{\mathbf{\Lambda}_\mathbf{x}} \underset{p\times 1}{\mathbf{\xi}_t} \underset{1\times T}{\mathbf{e}'}, \underset{A\times A}{\mathbf{\Theta}_\delta} \otimes \underset{T\times T}{\mathbf{I}_T}\right)$ dan berdasarkan Teorema B.2 (Lampiran 1) maka fungsi karakteristik dari $\boldsymbol{K}'$ adalah $\phi_{\mathbf{K}'}(\mathbf{Z}) = ets\left(\iota \underset{T\times A}{\mathbf{Z}_1'} \underset{A\times p}{\mathbf{\Lambda}_x} \underset{p\times 1}{\mathbf{\xi}_t} \underset{1\times T}{\mathbf{e}'} - 1/2 \underset{T\times A}{\mathbf{Z}_1'} \underset{A\times A}{\mathbf{\Theta}_\delta} \underset{A\times T}{\mathbf{Z}_t} \underset{T\times T}{\mathbf{I}_T}\right)$ dimana $\underset{T\times A}{\mathbf{Z}_1'} = \underset{T\times p}{\mathbf{Z}'} \underset{p\times A}{\mathbf{P}}$.

Berikut adalah penyederhanakan dari fungsi karakteristik dari $\boldsymbol{K}'$:

$$\phi_{\mathbf{K}'}(\mathbf{Z}) = ets\left(\iota \underset{T\times A}{\mathbf{Z}_1'} \underset{A\times p}{\mathbf{\Lambda_x}} \underset{p\times 1}{\xi_t} \underset{1\times T}{\mathbf{e}'} - 1/2 \underset{T\times A}{\mathbf{Z}_1'} \underset{A\times A}{\mathbf{\Theta}_\delta} \underset{A\times T}{\mathbf{Z}_t} \underset{T\times T}{\mathbf{I}_T} \right)$$ dengan menggantikan

$\underset{T\times A}{\mathbf{Z}_1'} = \underset{T\times p}{\mathbf{Z}'} \underset{p\times A}{P}$ maka

$$\phi_{\mathbf{K}'}(\mathbf{Z}) = ets\left(\iota \left(\underset{T\times p}{\mathbf{Z}'} \underset{p\times A}{\mathrm{P}} \right) \underset{A\times p}{\mathbf{\Lambda_x}} \underset{p\times 1}{\xi_t} \underset{1\times T}{\mathbf{e}'} - 1/2 \left(\underset{T\times p}{\mathbf{Z}'} \underset{p\times A}{\mathrm{P}} \right) \underset{A\times A}{\mathbf{\Theta}_\delta} \left(\underset{A\times p}{\mathbf{P}'} \underset{p\times T}{\mathbf{Z}} \right) \underset{T\times T}{\mathbf{I}_T} \right),$$

selanjutnya dengan menggantikan $\underset{A\times T}{\mathbf{P}} = \left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \underset{A\times p}{\mathbf{\Lambda_x}} \right)^{-1} \left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \right)$ maka akan berubah menjadi

$$\phi_{\mathbf{K}'}(\mathbf{Z}) = ets\left(\iota \underset{T\times p}{\mathbf{Z}'} \left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \underset{A\times p}{\mathbf{\Lambda_x}} \right) \right)^{-1} \left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \right) \underset{A\times p}{\mathbf{\Lambda_x}} \underset{p\times 1}{\xi_t} \underset{1\times T}{\mathbf{e}'} +$$

$$-1/2 \underset{T\times p}{\mathbf{Z}'} \left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \underset{A\times p}{\mathbf{\Lambda_x}} \right)^{-1} \left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \right) \underset{A\times A}{\mathbf{\Theta}_\delta} \left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \right)' \left(\left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \underset{A\times p}{\mathbf{\Lambda_x}} \right)' \right)^{-1} \underset{p\times T}{\mathbf{Z}} \underset{T\times T}{\mathbf{I}_T} \Bigg)$$

$$\phi_{\mathbf{K}'}(\mathbf{Z}) = ets\left(\iota \underset{T\times p}{\mathbf{Z}'} \underset{p\times 1}{\xi_t} \underset{1\times T}{\mathbf{e}'} - 1/2 \underset{T\times p}{\mathbf{Z}'} \left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \underset{A\times p}{\mathbf{\Lambda_x}} \right)^{-1} \underset{p\times A}{\mathbf{\Lambda_x}'} \left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \right)' \left(\left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \underset{A\times p}{\mathbf{\Lambda_x}} \right)' \right)^{-1} \underset{p\times T}{\mathbf{Z}} \underset{T\times T}{\mathrm{I}_T} \right)$$

$$\phi_{\mathbf{K}'}(\mathbf{Z}) = ets\left(\iota \underset{T\times p}{\mathbf{Z}'} \underset{p\times 1}{\xi_t} \underset{1\times T}{\mathbf{e}'} - 1/2 \underset{T\times p}{\mathbf{Z}'} \left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \underset{A\times p}{\mathbf{\Lambda_x}} \right)^{-1} \left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \underset{A\times p}{\mathbf{\Lambda_x}'} \right) \left(\left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \underset{A\times p}{\mathbf{\Lambda_x}} \right)' \right)^{-1} \underset{p\times T}{\mathbf{Z}} \underset{T\times T}{\mathrm{I}_T} \right)$$

Selanjutnya disederhanakan menjadi

$$\phi_{\mathbf{K}'}(\mathbf{Z}) = ets\left(\iota \underset{T\times p}{\mathbf{Z}'} \underset{p\times 1}{\xi_t} \underset{1\times T}{\mathbf{e}'} - 1/2 \underset{T\times p}{\mathbf{Z}'} \left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \underset{A\times p}{\mathbf{\Lambda_x}} \right)^{-1} \underset{p\times T}{\mathbf{Z}} \underset{T\times T}{\mathbf{I}_T} \right). \tag{3.5}$$

Persamaan (3.5) merupakan fungsi karakteristik dari $\boldsymbol{K}'$ dan berdasarkan Teorema B.2 (Lampiran 1) maka $\boldsymbol{K}'$ berdistribusi normal variat matrik dengan mean $\underset{p\times 1}{\xi_t} \underset{1\times T}{\mathbf{e}'}$ dan matrik kovarian $\left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \underset{A\times p}{\mathbf{\Lambda_x}} \right)^{-1} \otimes \underset{T\times T}{\mathbf{I}_T}$ serta dinotasikan $\underset{p\times T}{\boldsymbol{K}'} \sim N_{p,T}\left(\underset{p\times 1}{\xi_t} \underset{1\times T}{\mathbf{e}'}, \left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \underset{A\times p}{\mathbf{\Lambda_x}} \right)^{-1} \otimes \underset{T\times T}{\mathbf{I}_T} \right)$.

Berdasarkan Teorema B.1 (Lampiran 1) maka matrik K berdistribusi normal variat matrik yang dinotasikan sebagai berikut:

$$\underset{T\times p}{\boldsymbol{K}} \sim N_{p,T}\left(\underset{T\times 1}{\mathbf{e}'} \underset{1\times p}{\xi_t}, \underset{T\times T}{\mathbf{I}_T} \otimes \left(\underset{p\times A}{\mathbf{\Lambda_x}'} \underset{A\times A}{\mathbf{\Theta}_\delta^{-1}} \underset{A\times p}{\mathbf{\Lambda_x}} \right)^{-1} \right). \tag{3.6}$$

3.2. Estimasi Variabel Laten Endogen

Persamaan model pengukuran dengan variabel laten endogen sebanyak 1 sebagaimana persamaan (2.4) adalah $\underset{B\times 1}{\mathbf{y}} = \underset{B\times 1}{\mathbf{\Lambda}_{\mathbf{y}}} \underset{1\times 1}{\eta} + \underset{B\times 1}{\boldsymbol{\varepsilon}^*}$, dengan B adalah jumlah indikator variabel laten endogen, $\boldsymbol{\varepsilon}^*$ adalah *error* dimana $\boldsymbol{\varepsilon}^* \sim N\left(\mathbf{0}, \mathbf{\Theta}_{\varepsilon^*}\right)$ dengan $\mathbf{\Theta}_{\varepsilon^*}$ berukuran $B \times B$. Matrik kovarian dari kesalahan pengukuran variabel teramati $\mathbf{y}$ adalah $\underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}} = diag\left(\sigma^2_{\delta_1}, \sigma^2_{\delta_2}, \ldots, \sigma^2_{\delta_B}\right)$.

Persamaan model pengukuran dari variabel laten endogen dapat ditulis kembali dalam bentuk matrik sebagai berikut:

$$\underset{B\times 1}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_B \end{bmatrix}} = \underset{B\times 1}{\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_B \end{bmatrix}} \underset{1\times 1}{[\eta]} + \underset{B\times 1}{\begin{bmatrix} \varepsilon_1^* \\ \varepsilon_2^* \\ \vdots \\ \varepsilon_B^* \end{bmatrix}}.$$

Distribusi matrik random pengamatan dari vektor variabel teramati $\mathbf{y}$ sebagaimana persamaan (2.4) dimana $\boldsymbol{\varepsilon}^* \sim N\left(\mathbf{0}, \mathbf{\Theta}_{\boldsymbol{\varepsilon}^*}\right)$ dengan asumsi bahwa nilai *loading* faktor $\mathbf{\Lambda}_y$ dan matrik varian *error* $\mathbf{\Theta}_{\varepsilon^*}$ adalah konstan akan diurai pada Proposisi 1.

Proposisi 1.

Jika $\mathbf{y} = \mathbf{\Lambda}_y \eta + \boldsymbol{\varepsilon}^*$ dimana $\boldsymbol{\varepsilon}^* \sim N\left(\mathbf{0}, \mathbf{\Theta}_{\varepsilon^*}\right)$ dengan asumsi $\mathbf{\Lambda}_y$ dan $\mathbf{\Theta}_{\boldsymbol{\varepsilon}^*}$ konstan, maka distribusi $\mathbf{y}$ adalah $\mathbf{y} \sim N\left(\mathbf{\Lambda}_{\mathbf{y}}\eta, \mathbf{\Theta}_{\varepsilon^*}\right)$.

Bukti Proposisi 1.

Distribusi $\mathbf{y}$ didapatkan sebagaimana pada Lemma 1 dengan variabel laten dibatasi sejumlah satu. Nilai ekspektasi dan varian berturut-turut adalah $E(\mathbf{y}) = E\left(\mathbf{\Lambda}_{\mathbf{y}}\eta + \boldsymbol{\varepsilon}^*\right) = \mathbf{\Lambda}_y \eta$ dan $\mathrm{var}(\mathbf{y}) = \mathrm{var}\left(\mathbf{\Lambda}_{\mathbf{y}}\eta + \boldsymbol{\varepsilon}^*\right) = \mathbf{\Theta}_{\varepsilon^*}$, sehingga distribusi $\mathbf{y}$ adalah $\underset{B\times 1}{\mathbf{y}} \sim N\left(\underset{B\times 1}{\mathbf{\Lambda}_{\mathbf{y}}} \underset{1\times 1}{\eta}, \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}}\right)$.

Proposisi 2.

Jika $\mathbf{y} = \boldsymbol{\Lambda}_{\mathbf{y}}\eta + \boldsymbol{\varepsilon}^*$ dimana $\boldsymbol{\varepsilon}^* \sim N\left(\mathbf{0}, \boldsymbol{\Theta}_{\varepsilon^*}\right)$ dengan asumsi $\boldsymbol{\Lambda}_{\mathbf{y}}$ dan $\boldsymbol{\Theta}_{\varepsilon*}$ konstan, maka hasil estimasi η dengan menggunakan metode WLS adalah $\sum_{t=1}^{T}\hat{\eta}_t = \left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\boldsymbol{\varepsilon}^*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon^*}^{-1}\sum_{t=1}^{T}\mathbf{y}_t$.

Bukti Proposisi 2.

Hasil estimasi η sebagaimana Teorema 1, akan tetapi dibatasi variabel laten sejumlah satu. Berdasarkan Proposisi 1 distribusi $\mathbf{y}$ adalah $\mathbf{y} \sim N\left(\boldsymbol{\Lambda}_{\mathbf{y}}\eta, \boldsymbol{\Theta}_{\varepsilon^*}\right)$. Misalkan diberikan T sampel random dari variabel random $\mathbf{y}$ $\left(y_{11}, y_{12}, \ldots, y_{1T}, y_{21}, y_{22}, \ldots, y_{2T}, y_{B1}, y_{B2}, \ldots y_{BT}, \eta_1, \eta_2, \ldots, \eta_T\right)$ dengan $t = 1,2,3,\ldots,T$ maka $\mathbf{y}_t \sim N\left(\boldsymbol{\Lambda}_{\mathbf{y}}\eta_t, \boldsymbol{\Theta}_{\varepsilon^*}\right)$. Fungsi probabilitas dari $\mathbf{y}_t$ adalah $f\left(\mathbf{y}_t\right) = \left(2\pi\right)^{-\frac{B}{2}}\left|\boldsymbol{\Theta}_{\varepsilon^*}\right|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}\left(\mathbf{y}_t - \boldsymbol{\Lambda}_{\mathbf{y}}\eta_t\right)\boldsymbol{\Theta}_{\varepsilon^*}^{-1}\left(\mathbf{y}_t - \boldsymbol{\Lambda}_{\mathbf{y}}\eta_t\right)\right\}$

Fungsi *likelihood*nya adalah $L\left(\boldsymbol{\eta}_{\mathbf{t}}, \boldsymbol{\Theta}_{\varepsilon^*}\right) = \prod_{t=1}^{T} f\left(\mathbf{y}_t\right)$

$$L\left(\boldsymbol{\eta}_{\mathbf{t}}, \boldsymbol{\Theta}_{\varepsilon^*}\right) = \left(2\pi\right)^{-\frac{BT}{2}}\left|\boldsymbol{\Theta}_{\varepsilon^*}\right|^{-\frac{T}{2}}\exp\left\{-\frac{1}{2}\sum_{t=1}^{T}\left(\mathbf{y}_{\mathbf{t}} - \boldsymbol{\Lambda}_{\mathbf{y}}\eta_t\right)\right\}$$

$$L\left(\boldsymbol{\eta}_{\mathbf{t}}, \boldsymbol{\Theta}_{\varepsilon^*}\right) = \left(2\pi\right)^{-\frac{BT}{2}}\left|\boldsymbol{\Theta}_{\varepsilon^*}\right|^{-\frac{T}{2}}\exp\left\{-\frac{1}{2}Q_y\right\}$$

dengan . $\underset{1\times 1}{Q_y} = \sum_{t=1}^{T}\left(\underset{B\times 1}{\mathbf{y}_t} - \underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\underset{1\times 1}{\eta}\right)\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\left(\underset{B\times 1}{\mathbf{y}_t} - \underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\underset{1\times 1}{\eta}\right)$

Variabel laten $\boldsymbol{\eta}$ diestimasi menggunakan metode WLS dengan optimasi $L\left(\eta_t, \boldsymbol{\Theta}_{\varepsilon^*}\right)$. Maksimum $L\left(\eta_t, \boldsymbol{\Theta}_{\varepsilon^*}\right) \Leftrightarrow \min Q_y$.

$$Q_y = \sum_{t=1}^{T}\left(\mathbf{y}_t - \boldsymbol{\Lambda}_y\eta_t\right)\boldsymbol{\Theta}_{\varepsilon^*}^{-1}\left(\mathbf{y}_t - \boldsymbol{\Lambda}_y\eta_t\right)$$

$$\underset{1\times 1}{Q_y} = \sum_{t=1}^{T}\underset{1\times B}{\mathbf{y}_{\mathbf{t}}^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{y_t} - 2\sum_{t=1}^{T}\underset{1\times 1}{\eta_t^{'}}\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\mathbf{y}_{\mathbf{t}}} + \sum_{t=1}^{T}\underset{1\times 1}{\eta_t^{'}}\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\underset{1\times 1}{\eta_t}$$

Selanjutnya dicari turunan parsial dari Q_y terhadap $\boldsymbol{\eta}$ adalah:

$$\frac{\partial \mathbf{Q}_y}{\partial \boldsymbol{\eta}_t} = -2\sum_{t=1}^{T}\underset{1\times B}{\boldsymbol{\Lambda}_y^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\mathbf{y}_t} + 2\sum_{t=1}^{T}\underset{1\times B}{\boldsymbol{\Lambda}_y^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\boldsymbol{\Lambda}_y}\underset{1\times 1}{\eta_t} .$$

Jika hasil turunan parsial dari Q_y terhadap η disamakan dengan nol, maka didapatkan nilai skor faktor dari η, yaitu:

$$\sum_{t=1}^{T} \underset{1\times B}{\mathbf{\Lambda}_y^{'}} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\mathbf{\Lambda}_y} \underset{1\times 1}{\hat{\eta}_t} = \sum_{t=1}^{T} \underset{1\times B}{\mathbf{\Lambda}_y^{'}} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\mathbf{y}_t}$$

$$\left(\underset{1\times B}{\mathbf{\Lambda}_y^{'}} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\mathbf{\Lambda}_y} \right) \sum_{t=1}^{T} \underset{1\times 1}{\hat{\eta}_t} = \underset{1\times B}{\mathbf{\Lambda}_y^{'}} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \sum_{t=1}^{T} \underset{B\times 1}{\mathbf{y}_t}$$

$$\sum_{t=1}^{T} \underset{1\times 1}{\hat{\eta}_t} = \left(\underset{1\times B}{\mathbf{\Lambda}_{\mathbf{y}}^{'}} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\mathbf{\Lambda}_{\mathbf{y}}} \right)^{-1} \underset{1\times B}{\mathbf{\Lambda}_{\mathbf{y}}^{'}} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \sum_{t=1}^{T} \underset{B\times 1}{\mathbf{y}_t} . \qquad (3.7)$$

Distribusi matrik random pengamatan dari vektor variabel teramati $\mathbf{y}$ sebagaimana persamaan (2.4) dimana $\boldsymbol{\varepsilon}^* \sim N\left(\mathbf{0}, \mathbf{\Theta}_{\varepsilon^*}\right)$ dengan asumsi bahwa nilai *loading* faktor $\mathbf{\Lambda}_y$ dan matrik varian *error* $\mathbf{\Theta}_{\varepsilon^*}$ adalah konstan akan diurai pada Proposisi 3.

Proposisi 3.

Jika $\mathbf{y} = \mathbf{\Lambda}_y \eta + \boldsymbol{\varepsilon}^*$ dimana $\boldsymbol{\varepsilon}^* \sim N\left(\mathbf{0}, \mathbf{\Theta}_{\varepsilon^*}\right)$ dengan asumsi $\mathbf{\Lambda}_{\mathbf{y}}$ dan $\mathbf{\Theta}_{\varepsilon^*}$ konstan, maka distribusi matrik random pengamatan dari vektor variabel laten teramati $\mathbf{y}$ adalah $\mathbf{Y} \sim N_{B,T}\left(\mathbf{\Lambda}_y \eta_t \mathbf{e}^{'}, \mathbf{\Theta}_{\varepsilon^*} \otimes \mathbf{I}_T\right)$ dimana $\mathbf{e}_{T\times 1} = \left(1, \ldots, 1\right)$.

Bukti Proposisi 3.

Distribusi $\mathbf{Y}$ didapatkan sebagaimana langkah-langakh pada Teorema 2. Persamaan (3.7) dituliskan kembali dalam bentuk matrik adalah sebagai berikut:

$$\underset{1\times 1}{\left(\hat{\eta}_1 + \hat{\eta}_2 + \ldots + \hat{\eta}_T \right)} = \left(\underset{1\times B}{\mathbf{\Lambda}_{\mathbf{y}}^{'}} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\mathbf{\Lambda}_{\mathbf{y}}} \right)^{-1} \underset{1\times B}{\mathbf{\Lambda}_{\mathbf{y}}^{'}} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\begin{pmatrix} y_{11} + y_{12} + \ldots + y_{1T} \\ y_{21} + y_{22} + \ldots + y_{2T} \\ \vdots \\ y_{B1} + y_{B2} + \ldots + y_{BT} \end{pmatrix}} \qquad (3.8)$$

.

Jika persamaan (3.8) dibuat persamaan dalam bentuk matrik yang memuat elemen y dan $\hat{\eta}$ adalah

$$\left(\hat{\eta}_1+\hat{\eta}_2+\ldots+\hat{\eta}_T\right)_{1\times 1}=\left(\underset{1\times B}{\mathbf{\Lambda}_{\mathbf{y}}^{'}}\underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\mathbf{\Lambda}_y}\right)^{-1}\underset{1\times B}{\mathbf{\Lambda}_{\mathbf{y}}^{'}}\underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times T}{\begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1T} \\ y_{21} & y_{22} & \cdots & y_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ y_{B1} & y_{B12} & \cdots & y_{BT}\end{pmatrix}} \quad (3.9)$$

Pada persamaan (3.9) dimisalkan $\mathbf{Y}=\underset{B\times T}{\begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1T} \\ y_{21} & y_{22} & \cdots & y_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ y_{B1} & y_{B12} & \cdots & y_{BT}\end{pmatrix}}$

dimana **Y** adalah matrik random pengamatan dari vektor variabel laten teramati $\mathbf{y}$ dengan ukuran $B\times T$. Berdasarkan Definisi B.1 (Lampiran 1) dengan maka distribusi matrik $\mathbf{Y}^{'}$ adalah $\mathbf{Y}^{'}\sim N_{B,T}\left(\underset{T\times 1}{\mathbf{e}}\left(\underset{B\times 1}{\mathbf{\Lambda}_{\mathbf{y}}}\underset{1\times 1}{\eta_t}\right),\underset{T\times T}{\mathbf{I}_T}\otimes\underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}}\right)$ dimana $\mathbf{e}_{T\times 1}=(1,\ldots,1)^{'}$. Berdasarkan Teorema B.1 (Lampiran 1) maka distribusi **Y** adalah $\mathbf{Y}\sim N_{B,T}\left(\underset{B\times 1}{\mathbf{\Lambda}_{\mathbf{y}}}\underset{1\times 1}{\eta_t}\underset{1\times T}{\mathbf{e}^{'}},\underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}}\otimes\underset{T\times T}{\mathbf{I}_T}\right)$.

Pada persamaan (3.9) dengan memisalkan $\boldsymbol{l}^{'}=\underset{1\times T}{\left(\hat{\eta}_1\hat{\eta}_2\ldots\hat{\eta}_T\right)}$ maka persamaan (3.9) dapat disederhanakan menjadi

$$\underset{1\times T}{\boldsymbol{l}^{'}}=\left(\underset{1\times B}{\mathbf{\Lambda}_{\mathbf{y}}^{'}}\underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\mathbf{\Lambda}_{\mathbf{y}}}\right)^{-1}\underset{1\times B}{\mathbf{\Lambda}_y^{'}}\underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times T}{\mathbf{Y}}\;. \quad (3.10)$$

Distribusi $\boldsymbol{l}$ akan dipaparkan pada Proposisi 4 berikut.

Proposisi 4.

Jika distribusi matrik random pengamatan dari vektor variabel laten teramati $\mathbf{y}$ adalah $\mathbf{Y}\sim N_{B.T}\left(\mathbf{\Lambda}_{\mathbf{y}}\eta_t\mathbf{e}^{'},\mathbf{\Theta}_{\varepsilon^*}\otimes\mathbf{I}_T\right)$ dan $\boldsymbol{l}^{'}=\left(\mathbf{\Lambda}_{\mathbf{y}}^{'}\mathbf{\Theta}_{\varepsilon^*}^{-1}\mathbf{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{\Lambda}_{\mathbf{y}}^{'}\mathbf{\Theta}_{\varepsilon^*}^{-1}\mathbf{Y}$ maka distribusi dari $\boldsymbol{l}$ adalah $\boldsymbol{l}\sim N\left(\mathbf{e}\eta_t,\left(\mathbf{\Lambda}_{\mathbf{y}}^{'}\mathbf{\Theta}_{\varepsilon^*}^{-1}\mathbf{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right)$

Bukti Proposisi 4:

Definisi fungsi karakteristik dari matrik random Y adalah $\phi_Y(\mathbf{Z})=E\left[etr\left(t\mathbf{YZ}^{'}\right)\right]$ dengan $t=\sqrt{-1}$. Jika bagian dari persamaan

(3.10) dimisalkan $\left(\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1}\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}=\mathbf{R}$ maka persamaan (3.10) dapat disederhanakan menjadi $\underset{1\times T}{\boldsymbol{l}^{'}}=\underset{1\times B}{\mathbf{R}}\underset{B\times T}{\mathbf{Y}}$. Berdasarkan Teorema C.1 (Lampiran 1) maka fungsi karakteristik dari $\boldsymbol{l}^{'}$ dapat diubah menjadi $\phi_{l'}(\mathbf{Z})=E\left[etr\left(\iota\left(\underset{B\times T}{\mathbf{Y}}\underset{T\times 1}{\mathbf{Z}}\right)\underset{1\times B}{\mathbf{R}}\right)\right]=E\left[etr\left(\iota\underset{B\times T}{\mathbf{Y}}\left(\underset{T\times 1}{\mathbf{Z}^{'}}\underset{1\times B}{\mathbf{R}}\right)\right)\right]$. Misalkan $\underset{T\times B}{\mathbf{Z}_2^{'}}=\underset{T\times 1}{\mathbf{Z}^{'}}\underset{1\times B}{\mathbf{R}}$ maka fungsi karakteristik dari $l^{'}$ dapat diubah kembali menjadi $\phi_{l'}(\mathbf{Z})=E\left[etr\left(\iota\underset{B\times T}{\mathbf{Y}}\underset{T\times B}{\mathbf{Z}_2^{'}}\right)\right]$. Distribusi $\mathbf{Y}$ adalah $\mathbf{Y}\sim N_{B,T}\left(\underset{B\times 1}{\boldsymbol{\Lambda}_y}\underset{1\times 1}{\eta_t}\underset{1\times T}{\mathbf{e}^{'}},\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}}\otimes\underset{T\times T}{\mathbf{I}_T}\right)$ dan berdasarkan Teorema B.2 (Lampiran 1) maka fungsi karakteristik dari $l^{'}$ adalah $\phi_{l'}(\mathbf{Z})=etr\left(\iota\underset{T\times B}{\mathbf{Z}_2^{'}}\underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\underset{1\times 1}{\eta_t}\underset{1\times T}{\mathbf{e}^{'}}-\frac{1}{2}\underset{T\times B}{\mathbf{Z}_2^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}}\underset{B\times T}{\mathbf{Z}_2}\underset{T\times T}{\mathbf{I}_T}\right)$ dimana $\underset{T\times B}{\mathbf{Z}_2^{'}}=\underset{T\times 1}{\mathbf{Z}^{'}}\underset{1\times B}{\mathbf{R}}$.

Berikut adalah penyederhanaan dari fungsi karakteristik dari $l^{'}$:

Pada $\phi_{l'}(\mathbf{Z})=etr\left(\iota\underset{T\times B}{\mathbf{Z}_2^{'}}\underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\underset{1\times 1}{\eta_t}\underset{1\times T}{\mathbf{e}^{'}}-1/2\underset{T\times B}{\mathbf{Z}_2^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}}\underset{B\times T}{\mathbf{Z}_2}\underset{T\times T}{\mathbf{I}_T}\right)$ dengan menggantikan $\underset{T\times B}{\mathbf{Z}_2^{'}}=\underset{T\times 1}{\mathbf{Z}^{'}}\underset{1\times B}{\mathbf{R}}$ maka

$$\phi_{l'}(\mathbf{Z})=etr\left(\iota\left(\underset{T\times B}{\mathbf{Z}_2^{'}}\underset{1\times B}{\mathbf{R}}\right)\underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\underset{1\times 1}{\eta_t}\underset{1\times T}{\mathbf{e}^{'}}-1/2\left(\underset{T\times 1}{\mathbf{Z}^{'}}\underset{1\times B}{\mathbf{R}}\right)\underset{B\times B}{\boldsymbol{\Theta}_{\boldsymbol{\varepsilon}^*}}\left(\underset{B\times 1}{\mathbf{R}^{'}}\underset{1\times T}{\mathbf{Z}}\right)\underset{T\times T}{\mathbf{I}_T}\right),$$

selanjutnya dengan menggantikan $\underset{1\times B}{\mathbf{R}}=\left(\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1}\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}$ maka akan berubah menjadi

$$\phi_{l'}(\mathbf{Z})=etr\left(\iota\underset{T\times B}{\mathbf{Z}_2^{'}}\left(\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1}\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\underset{1\times 1}{\eta_t}\underset{1\times T}{\mathbf{e}^{'}}-1/2\underset{T\times 1}{\mathbf{Z}^{'}}\left(\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1}\times\right.$$

$$\left.\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}}\left(\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\right)\underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\left(\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1}\underset{1\times T}{\mathbf{Z}}\underset{T\times T}{\mathbf{I}_{\mathrm{T}}}\right)$$

$$\phi_{l'}(\mathbf{Z}) = etr\left(\iota \underset{T\times B}{\mathbf{Z}_2'} \underset{1\times 1}{\eta_t} \underset{1\times T}{\mathbf{e}'} - 1/2 \underset{\mathbf{T\times 1}}{\mathbf{Z}'} \left(\underset{1\times B}{\mathbf{\Lambda}_{\mathbf{y}}'} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\mathbf{\Lambda}_{\mathbf{y}}}\right)^{-1} \left(\underset{1\times B}{\mathbf{\Lambda}_{\mathbf{y}}'} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\mathbf{\Lambda}_{\mathbf{y}}}\right)' \times \right.$$

$$\left.\left(\left(\underset{1\times B}{\mathbf{\Lambda}_{\mathbf{y}}'} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\mathbf{\Lambda}_{\mathbf{y}}}\right)^{-1}\right)' \underset{1\times T}{\mathbf{Z}} \underset{T\times T}{\mathbf{I}_T}\right)$$

Selanjutnya disederhanakan menjadi:

$$\phi_{l'}(\mathbf{Z}) = etr\left(\iota \underset{T\times B}{\mathbf{Z}_2'} \underset{1\times 1}{\eta_t} \underset{1\times T}{\mathbf{e}'} - 1/2 \underset{T\times 1}{\mathbf{Z}'} \left(\underset{1\times B}{\mathbf{\Lambda}_{\mathbf{y}}'} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\mathbf{\Lambda}_{\mathbf{y}}}\right)^{-1} \underset{1\times T}{\mathbf{Z}} \underset{T\times T}{\mathbf{I}_T}\right). \tag{3.11}$$

Persamaan (3.11) merupakan fungsi karakteristik dari l' dan berdasarkan Teorema B.2 (Lampiran 1) maka l' berdistribusi normal variat matrik dengan mean $\underset{1\times 1}{\boldsymbol{\eta}_t} \underset{1\times T}{\mathbf{e}'}$ dan matrik kovarian $\left(\underset{1\times B}{\mathbf{\Lambda}_{\mathbf{y}}'} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\mathbf{\Lambda}_{\mathbf{y}}}\right) \otimes \underset{T\times T}{\mathbf{I}_T}$ atau $\left(\underset{1\times B}{\mathbf{\Lambda}_{\mathbf{y}}'} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\mathbf{\Lambda}_{\mathbf{y}}}\right)^{-1} \otimes \underset{T\times T}{\mathbf{I}_T}$ serta dinotasikan sebagai $\underset{1\times T}{l'} \sim N_T\left(\underset{1\times 1}{\eta_t} \underset{1\times T}{\mathbf{e}'}, \left(\underset{1\times B}{\mathbf{\Lambda}_{\mathbf{y}}'} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\mathbf{\Lambda}_{\mathbf{y}}}\right)^{-1} \underset{T\times T}{\mathbf{I}_T}\right)$. Berdasarkan Teorema B.1 (Lampiran 1) maka matrik l berdistribusi normal variat matrik yang dinotasikan sebagai berikut:

$$\underset{T\times 1}{l} \sim N_T\left(\underset{T\times 1}{\mathbf{e}} \underset{1\times 1}{\eta_t}, \left(\underset{1\times B}{\mathbf{\Lambda}_{\mathbf{y}}'} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\mathbf{\Lambda}_{\mathbf{y}}}\right)^{-1} \underset{T\times T}{\mathbf{I}_T}\right). \tag{3.12}$$

BAB 4 DISTRIBUSI *ERROR* MODEL SAR-SEM DAN SERM-SEM

4.1. Distribusi *Error* dari Model SAR-SEM

Model SAR-SEM sebagaimana pada persamaan (2.16) dituliskan kembali dalam bentuk persamaan sebagai berikut:

$$
\begin{aligned}
l_1 &= \beta_0 + k_{11}\beta_1 + k_{12}\beta_2 + \dots k_{1p}\beta_p + \lambda w_{11} l_1 + \lambda w_{12} l_2 + \dots \lambda w_{1T} l_T + \varepsilon_1 \\
l_2 &= \beta_0 + k_{21}\beta_1 + k_{22}\beta_2 + \dots k_{2p}\beta_p + \lambda w_{21} l_1 + \lambda w_{22} l_2 + \dots \lambda w_{2T} l_T + \varepsilon_2 \\
&\vdots \\
l_T &= \beta_0 + k_{T1}\beta_1 + k_{T2}\beta_2 + \dots k_{Tp}\beta_p + \lambda w_{T1} l_1 + \lambda w_{T2} l_2 + \dots \lambda w_{TT} l_T + \varepsilon_T
\end{aligned}
$$

atau persamaan dalam matriknya adalah

$$
\underset{T\times 1}{\begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_T \end{pmatrix}} = \underset{T\times(p+1)}{\begin{pmatrix} 1 & k_{11} & k_{12} & \dots & k_{1p} \\ 1 & k_{21} & k_{22} & \dots & k_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & k_{T1} & k_{T2} & \dots & k_{Tp} \end{pmatrix}} + \lambda \underset{T\times T}{\begin{pmatrix} w_{11} & w_{12} & \dots & w_{1T} \\ w_{21} & w_{22} & \dots & w_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ w_{T1} & w_{T2} & \dots & w_{TT} \end{pmatrix}} \underset{T\times 1}{\begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_T \end{pmatrix}} + \underset{T\times 1}{\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_T \end{pmatrix}}
$$

dimana $\boldsymbol{K}$ berdistribusi sebagaimana persamaan (3.4) dan $\boldsymbol{l}$ berdistribusi sebagaimana persamaan (3.12).

Model SAR-SEM sebagaimana pada persamaan (2.16) dapat ditulis kembali $\boldsymbol{l} = (\mathbf{I} - \lambda\mathbf{W})^{-1}\mathbf{K}\boldsymbol{\beta} + (\mathbf{I} - \lambda\mathbf{W})^{-1}\boldsymbol{\varepsilon}$ dimana ini merupakan model regresi spasial dengan memandang variabel $\boldsymbol{l}$ merupakan variabel respon dan $\boldsymbol{K}$ variabel bebas yang bersifat random, sehingga fungsi dari $\boldsymbol{l}$ adalah $f(\boldsymbol{l} \mid \boldsymbol{K})$, oleh karena itu variabel $\boldsymbol{K}$ bukan lagi bersifat random tetapi tetap. *Error* pada persamaan tersebut adalah $\underset{T\times 1}{\boldsymbol{\varepsilon}} = \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}$.

Distribusi *error* didapatkan berdasarkan nilai ekspektasi dan varians:

$$E(\varepsilon) = E\left[\left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right]$$

$$= \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)E\left(\underset{T\times 1}{\boldsymbol{l}}\right) - \underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}$$

$$= \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)\underset{T\times 1}{\mathbf{e}}\underset{1\times 1}{\boldsymbol{\eta}} - \underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}$$

Nilai variansnya adalah:

$$\text{var}(\boldsymbol{\varepsilon}) = \text{var}\left[\left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right]$$

$$\text{var}(\boldsymbol{\varepsilon}) = \text{var}\left[\left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)\underset{T\times 1}{\boldsymbol{l}}\right]$$

$$\text{var}(\boldsymbol{\varepsilon}) = \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)\left(\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1}\left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)^{'}$$

sehingga distribusi *error* model SAR-SEM adalah

$$\boldsymbol{\varepsilon} \sim N_{T,1}\left(\left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)\underset{T\times 1}{\mathbf{e}}\underset{1\times 1}{\boldsymbol{\eta}_l} - \underset{T\times(p+1)}{K}\underset{(p+1)\times 1}{\boldsymbol{\beta}}, \underset{T\times T}{\boldsymbol{\Theta}_\lambda}\right), \quad (4.1)$$

Dengan $\boldsymbol{\Theta}_\lambda = \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)\left(\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^{'}}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1}\left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)^{'}$ dengan asumsi variabel K tidak berkorelasi dengan *error* $\boldsymbol{\varepsilon}$ dan $\text{cov}(\boldsymbol{l},\boldsymbol{K}) \neq \mathbf{0}$.

4.2. Distribusi *Error* dari Model SERM-SEM

Model SERM-SEM sebagaimana pada persamaan (2.18) dengan $\boldsymbol{K}$ berdistribusi sebagaimana persamaan (3.4) dan $\boldsymbol{l}$ berdistribusi sebagaimana persamaan (3.12). Model SERM-SEM menggunakan matrik pembobot yamg sama dengan model SAR-SEM. Model tersebut dapat ditulis kembali $\underset{T\times 1}{\boldsymbol{l}} = \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}} + \left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right)^{-1}\underset{T\times 1}{\boldsymbol{\varepsilon}}$ dimana ini merupakan model regresi spasial dengan memandang variabel $\boldsymbol{l}$ merupakan variabel respon dan $\boldsymbol{K}$ variabel bebas yang bersifat random, sehingga fungsi dari $\boldsymbol{l}$ adalah $f(\boldsymbol{l} \mid \boldsymbol{K})$, oleh karena itu variabel $\boldsymbol{K}$ bukan lagi bersifat random tetapi tetap. *Error* pada persamaan tersebut adalah $\underset{T\times 1}{\boldsymbol{\varepsilon}} = \left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)$.

Distribusi *error* didapatkan berdasarkan nilai ekspektasi dan varians:

$$
\begin{aligned}
E(\varepsilon) &= E\left[\left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right] \\
&= \left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right) E\left(\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right) \\
&= \left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right)\left(E\left(\underset{T\times 1}{\boldsymbol{l}}\right) - E\left(\underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right) \\
&= \left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times 1}{\mathbf{e}} \underset{1\times 1}{\boldsymbol{\eta}} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)
\end{aligned}
$$

Nilai variansnya adalah:

$$
\operatorname{var}(\boldsymbol{\varepsilon}) = \operatorname{var}\left[\left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right]
$$

$$
\operatorname{var}(\boldsymbol{\varepsilon}) = \operatorname{var}\left[\left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right)\underset{T\times 1}{\boldsymbol{l}} - \left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right]
$$

$$
\operatorname{var}(\boldsymbol{\varepsilon}) = \operatorname{var}\left[\left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right)\underset{T\times 1}{\boldsymbol{l}}\right] - \operatorname{var}\left[\left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right]
$$

$$
\operatorname{var}(\boldsymbol{\varepsilon}) = \operatorname{var}\left[\left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right)\underset{T\times 1}{\boldsymbol{l}}\right]
$$

$$
\operatorname{var}(\boldsymbol{\varepsilon}) = \left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right)\left(\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}'} \underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1} \underset{T\times T}{\mathbf{I}} \left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right)'
$$

sehingga distribusi *error* adalah

$$
\boldsymbol{\varepsilon} \sim N_{T,1}\left(\left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times 1}{\mathbf{e}} \underset{1\times 1}{\boldsymbol{\eta}} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}}\right), \underset{T\times T}{\boldsymbol{\Theta}_{\rho}}\right), \qquad (4.2)
$$

Dengan $\boldsymbol{\Theta}_{\rho} = \left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right)\left(\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}'} \underset{B\times B}{\boldsymbol{\Theta}_{\boldsymbol{\varepsilon}^*}^{-1}} \underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1} \underset{T\times T}{\mathbf{I}} \left(\underset{T\times T}{\mathbf{I}} - \rho \underset{T\times T}{\mathbf{W}}\right)'$ dengan

asumsi variabel $\boldsymbol{K}$ tidak berkorelasi dengan *error* $\boldsymbol{\varepsilon}$ dan $\text{cov}(\boldsymbol{l}, \boldsymbol{K}) \neq \mathbf{0}$

.

BAB 5

ESTIMASI PARAMETER MODEL SAR-SEM DAN SERM-SEM

5.1. Estimasi Parameter Model SAR-SEM

Jika parameter pada model SAR-SEM sebagaimana pada persamaan (2.16) dengan distribusi *error* sebagaimana persamaan (4.1) diduga menggunakan metode OLS dan metode momen, maka akan didapatkan penduga yang bias dan tidak konsisten. Hal ini karena terdapat kasus bahwa variabel regressor $\mathbf{W}\boldsymbol{l}$ berkorelasi dengan *error* ε atau $\text{cov}\left[\mathbf{W}\boldsymbol{l},\boldsymbol{\varepsilon}\right] \neq 0$ (Bukti pada Lampiran 3).

Persamaan (2.16) disederhanakan menjadi $\underset{T\times 1}{\boldsymbol{l}} = \underset{T\times(p+2)}{\mathbf{Z}}\,\underset{(p+2)\times 1}{\boldsymbol{\delta}} + \underset{T\times 1}{\boldsymbol{\varepsilon}}$

dimana $\underset{T\times(p+2)}{\mathbf{Z}} = \left(\underset{T\times(p+1)}{\boldsymbol{K}} \;\middle|\; \underset{T\times T}{\mathbf{W}}\,\underset{T\times 1}{\boldsymbol{l}}\right)$ dan $\underset{(p+2)\times 1}{\boldsymbol{\delta}} = \left(\underset{1\times(p+1)}{\boldsymbol{\beta}^{'}} \;\middle|\; \underset{1\times 1}{\lambda}\right)^{'}$.

Pada penelitian ini, $\boldsymbol{\delta}$ diduga menggunakan metode 2SLS, yaitu melalui metode OLS dengan 2 langkah sebagai berikut:

a. Pendugaan menggunakan metode 2SLS membutuhkan variabel intrumen $\mathbf{H}$, yaitu gabungan antara matrik $\boldsymbol{K}$ dan matrik $\mathbf{W}\boldsymbol{K}$ atau dituliskan $\underset{T\times 2(p+1)}{\mathbf{H}} = \left(\underset{T\times(p+1)}{\boldsymbol{K}} \;\middle|\; \underset{T\times T}{\mathbf{W}}\,\underset{T\times(p+1)}{\boldsymbol{K}}\right)$. Variabel instrumen $\underset{T\times 2(p+1)}{\mathbf{H}} = \left(\underset{T\times(p+1)}{\boldsymbol{K}} \;\middle|\; \underset{T\times T}{\mathbf{W}}\,\underset{T\times(p+1)}{\boldsymbol{K}}\right)$ adalah valid (Bukti pada Lampiran 4).

b. Melakukan regresi *ordinary least square* (OLS) antara variabel $\mathbf{Z}$ dengan variabel instrumen $\mathbf{H}$

   $\mathbf{Z} = \mathbf{H}\mathbf{H}^{-1}\left(\mathbf{H}^{'}\right)^{-1}\mathbf{H}^{'}\mathbf{Z} + \boldsymbol{\varepsilon}$ dengan memisalkan $\mathbf{Z} = \mathbf{R} + \boldsymbol{\varepsilon}$. Turunan pertama terhadap $\mathbf{Z}$ adalah $\underset{T\times(p+2)}{\hat{\mathbf{Z}}} = \underset{T\times(2p+1)}{\mathbf{H}}\left(\underset{(2p+1)\times T}{\mathbf{H}^{'}}\,\underset{T\times(2p+1)}{\mathbf{H}}\right)^{-1}\underset{(2p+1)\times T}{\mathbf{H}^{'}}\,\underset{T\times(p+2)}{\mathbf{Z}}$. Dengan memisalkan nilai $\underset{T\times T}{\mathbf{P_H}} = \underset{T\times(2p+1)}{\mathbf{H}}\left(\underset{(2p+1)\times T}{\mathbf{H}'}\underset{T\times(2p+1)}{\mathbf{H}}\right)^{-1}\underset{(2p+1)\times T}{\mathbf{H}'}$ maka didapatkan sifat-sifat dari $\mathbf{P_H}$ bersifat simetrik $(\mathbf{P_H} = \mathbf{P_H}')$ dan idempoten $\left((\mathbf{P_H})^2 = \mathbf{P_H}\times\mathbf{P_H} = \mathbf{P_H}\right)$.

Matrik $\mathbf{P_H}$ adalah matrik simetrik karena $\mathbf{P_H}' = \left(\mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'\right)' = \mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}' = \mathbf{P_H}$. Dan matrik $\mathbf{P_H}$ adalah matrik idempoten karena

$$(\mathbf{P_H})^2 = \mathbf{P_H}\mathbf{P_H} = \left[\mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'\right]\left[\mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'\right]$$

$$= \mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}(\mathbf{H}'\mathbf{H})(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'$$

$$= \mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}' = \mathbf{P_H}$$

Karena $\underset{T\times 2(p+1)}{\mathbf{H}} = \left(\underset{T\times(p+1)}{\boldsymbol{K}} \;\middle|\; \underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}\right)$ maka $\mathbf{P_H}$ dapat dituliskan secara lengkap sebagai berikut:

$$\underset{T\times T}{\mathbf{P_H}} = \underset{T\times(2p+1)}{\left(\underset{T\times(p+1)}{\boldsymbol{K}} \;\middle|\; \underset{T\times T}{\mathbf{W}}\underset{T\times p}{\boldsymbol{K}}\right)}\left[\underset{(2p+1)\times T}{\left(\underset{T\times(p+1)}{\boldsymbol{K}} \;\middle|\; \underset{T\times T}{\mathbf{W}}\underset{T\times p}{\boldsymbol{K}}\right)'}\underset{T\times(2p+1)}{\left(\underset{T\times(p+1)}{\boldsymbol{K}} \;\middle|\; \underset{T\times T}{\mathbf{W}}\underset{T\times p}{\boldsymbol{K}}\right)}\right]^{-1}\underset{(2p+1)\times T}{\left(\underset{T\times(p+1)}{\boldsymbol{K}} \;\middle|\; \underset{T\times T}{\mathbf{W}}\underset{T\times p}{\boldsymbol{K}}\right)'}$$

$$= \underset{T\times(2p+1)}{\left(\underset{T\times(p+1)}{\boldsymbol{K}} \;\middle|\; \underset{T\times T}{\mathbf{W}}\underset{T\times p}{\boldsymbol{K}}\right)}\left[\underset{(p+2)\times T}{\left(\frac{\underset{(p+1)\times T}{\boldsymbol{K}'}}{\underset{p\times T}{(\mathbf{W}\boldsymbol{K})'}}\right)}\underset{T\times(2p+1)}{\left(\underset{T\times(p+1)}{\boldsymbol{K}} \;\middle|\; \underset{T\times T}{\mathbf{W}}\underset{T\times p}{\boldsymbol{K}}\right)}\right]^{-1}\underset{(p+2)\times T}{\left(\frac{\underset{(p+1)\times T}{\boldsymbol{K}'}}{\underset{p\times T}{(\mathbf{W}\boldsymbol{K})'}}\right)}$$

$$= \underset{T\times(2p+1)}{\left(\underset{T\times(p+1)}{\boldsymbol{K}} \;\middle|\; \underset{T\times T}{\mathbf{W}}\underset{T\times p}{\boldsymbol{K}}\right)}\left(\begin{array}{c|c}\underset{(p+1)\times(p+1)}{\boldsymbol{K}'\boldsymbol{K}} & \underset{(p+1)\times p}{\boldsymbol{K}'\mathbf{W}\boldsymbol{K}} \\ \hline \underset{p\times(p+1)}{(\mathbf{W}\boldsymbol{K})'\boldsymbol{K}} & \underset{p\times p}{(\mathbf{W}\boldsymbol{K})'\mathbf{W}\boldsymbol{K}}\end{array}\right)^{-1}\underset{(p+2)\times T}{\left(\frac{\underset{(p+1)\times T}{\boldsymbol{K}'}}{\underset{p\times T}{(\mathbf{W}\boldsymbol{K})'}}\right)}$$

selanjutnya terlebih dahulu akan dicari $\left(\begin{array}{c|c} \underset{(p+1)\times(p+1)}{\boldsymbol{K}'\boldsymbol{K}} & \underset{(p+1)\times p}{\boldsymbol{K}'\mathbf{W}\boldsymbol{K}} \\ \hline \underset{p\times(p+1)}{(\mathbf{W}\boldsymbol{K})'\boldsymbol{K}} & \underset{p\times p}{(\mathbf{W}\boldsymbol{K})'\mathbf{W}\boldsymbol{K}} \end{array}\right)^{-1}$

dengan memisalkan dengan memisalkan $\mathbf{A}_1 = \underset{(p+1)\times(p+1)}{\boldsymbol{K}'\boldsymbol{K}}$; $\mathbf{A}_2 = \underset{(p+1)\times p}{\boldsymbol{K}'\mathbf{W}\boldsymbol{K}}$; $\mathbf{A}_3 = \underset{p\times(p+1)}{(\mathbf{W}\boldsymbol{K})'\boldsymbol{K}}$; dan $\mathbf{A}_4 = \underset{p\times p}{(\mathbf{W}\boldsymbol{K})'\mathbf{W}\boldsymbol{K}}$.

Jika $\mathbf{A}_{E1} = \left[\mathbf{A}_1 - \mathbf{A}_2\mathbf{A}_4^{-1}\mathbf{A}_3\right]^{-1}$ maka

$$\underset{(p+1)\times(p+1)}{\mathbf{A}_{E1}} = \left[\underset{(p+1)\times(p+1)}{\boldsymbol{K}'\boldsymbol{K}} - \underset{(p+1)\times p}{\boldsymbol{K}'\mathbf{W}\boldsymbol{K}}\left(\underset{p\times p}{(\mathbf{W}\boldsymbol{K})'\mathbf{W}\boldsymbol{K}}\right)^{-1}\underset{p\times(p+1)}{(\mathbf{W}\boldsymbol{K})'\boldsymbol{K}}\right]^{-1}. \quad (5.1)$$

Jika $\mathbf{A}_{E2} = \left[-\mathbf{A}_{E1}\mathbf{A}_2\mathbf{A}_4^{-1}\right]$, maka

$$\underset{(p+1)\times p}{\mathbf{A}_{E2}} = -\left[\underset{(p+1)\times(p+1)}{\boldsymbol{K}'\boldsymbol{K}} - \underset{(p+1)\times p}{\boldsymbol{K}'\mathbf{W}\boldsymbol{K}}\left(\underset{p\times p}{(\mathbf{W}\boldsymbol{K})'\mathbf{W}\boldsymbol{K}}\right)^{-1}\underset{p\times(p+1)}{(\mathbf{W}\boldsymbol{K})'\boldsymbol{K}}\right]^{-1} \times \underset{(p+1)\times p}{\boldsymbol{K}'\mathbf{W}\boldsymbol{K}}\left(\underset{p\times p}{(\mathbf{W}\boldsymbol{K})'\mathbf{W}\boldsymbol{K}}\right)^{-1}. \quad (5.2)$$

Jika $\mathbf{A}_{E3} = \left[-\mathbf{A}_4^{-1}\mathbf{A}_3\mathbf{A}_{E1}\right]$, maka

$$\underset{p\times(p+1)}{\mathbf{A}_{E3}} = -\left(\underset{p\times p}{(\mathbf{W}\boldsymbol{K})'\mathbf{W}\boldsymbol{K}}\right)^{-1}\underset{p\times(p+1)}{(\mathbf{W}\boldsymbol{K})'\boldsymbol{K}} \times \left[\underset{(p+1)\times(p+1)}{\boldsymbol{K}'\boldsymbol{K}} - \underset{(p+1)\times p}{\boldsymbol{K}'\mathbf{W}\boldsymbol{K}}\left(\underset{p\times p}{(\mathbf{W}\boldsymbol{K})'\mathbf{W}\boldsymbol{K}}\right)^{-1}\underset{p\times(p+1)}{(\mathbf{W}\boldsymbol{K})'\boldsymbol{K}}\right]^{-1}. \quad (5.3)$$

Jika $\mathbf{A}_{E4} = \left[\mathbf{A}_4^{-1} - \mathbf{A}_4^{-1}\mathbf{A}_3\mathbf{A}_{E2}\right]$, maka

$$\underset{p\times p}{\mathbf{A}_{E4}} = \left(\underset{p\times p}{(\mathbf{W}\boldsymbol{K})'\mathbf{W}\boldsymbol{K}}\right)^{-1} + \left(\underset{p\times p}{(\mathbf{W}\boldsymbol{K})'\mathbf{W}\boldsymbol{K}}\right)^{-1}\underset{p\times(p+1)}{(\mathbf{W}\boldsymbol{K})'\boldsymbol{K}} \times \left(\underset{(p+1)\times(p+1)}{\boldsymbol{K}'\boldsymbol{K}} - \underset{(p+1)\times p}{\boldsymbol{K}'\mathbf{W}\boldsymbol{K}}\left(\underset{p\times p}{(\mathbf{W}\boldsymbol{K})'\mathbf{W}\boldsymbol{K}}\right)^{-1}\underset{p\times(p+1)}{(\mathbf{W}\boldsymbol{K})'\boldsymbol{K}}\right)^{-1}\underset{(p+1)\times p}{\boldsymbol{K}'\mathbf{W}\boldsymbol{K}}\left(\underset{p\times p}{(\mathbf{W}\boldsymbol{K})'\mathbf{W}\boldsymbol{K}}\right)^{-1}. \quad (5.4)$$

Nilai P_H menjadi sebagai berikut:

$$\underset{T\times T}{\mathbf{P_H}} = \underset{T\times(2p+1)}{\left(\underset{T\times(p+1)}{\boldsymbol{K}} \;\middle|\; \underset{T\times T}{\mathbf{W}}\underset{T\times p}{\boldsymbol{K}}\right)} \left[\underset{(p+2)\times T}{\left(\begin{array}{c} \underset{(p+1)\times T}{\boldsymbol{K}'} \\ \hline \underset{p\times T}{(\mathbf{W}\boldsymbol{K})'} \end{array}\right)} \underset{T\times(2p+1)}{\left(\underset{T\times(p+1)}{\boldsymbol{K}} \;\middle|\; \underset{T\times T}{\mathbf{W}}\underset{T\times p}{\boldsymbol{K}}\right)} \right]^{-1} \underset{(p+2)\times T}{\left(\begin{array}{c} \underset{(p+1)\times T}{\boldsymbol{K}'} \\ \hline \underset{p\times T}{(\mathbf{W}\boldsymbol{K})'} \end{array}\right)}$$

$$\underset{T\times T}{\mathbf{P_H}} = \underset{T\times(2p+1)}{\left(\underset{T\times(p+1)}{\boldsymbol{K}} \;\middle|\; \underset{T\times T}{\mathbf{W}}\underset{T\times p}{\boldsymbol{K}}\right)} \underset{(2p+1)\times(p+2)}{\left(\begin{array}{c|c} \underset{(p+1)\times(p+1)}{\mathbf{A}_{E1}} & \underset{(p+1)\times p}{\mathbf{A}_{E2}} \\ \hline \underset{p\times(p+1)}{\mathbf{A}_{E3}} & \underset{p\times p}{\mathbf{A}_{E4}} \end{array}\right)} \underset{(p+2)\times T}{\left(\begin{array}{c} \underset{(p+1)\times T}{\boldsymbol{K}'} \\ \hline \underset{p\times T}{(\mathbf{W}\boldsymbol{K})'} \end{array}\right)}$$

$$\underset{T\times T}{\mathbf{P_H}} = \underset{T\times T}{\left(\boldsymbol{K}\mathbf{A}_{E1}\boldsymbol{K}'\right)} + \underset{T\times T}{\left(\mathbf{W}\boldsymbol{K}\mathbf{A}_{E3}\boldsymbol{K}'\right)} + \underset{T\times T}{\left(\boldsymbol{K}\mathbf{A}_{E2}(\mathbf{W}\boldsymbol{K})'\right)} + \underset{T\times T}{\left(\mathbf{W}\boldsymbol{K}\mathbf{A}_{E4}(\mathbf{W}\boldsymbol{K})'\right)}. \qquad (5.5)$$

dimana A_{E1}, A_{E2},A_{E3}, dan A_{E4} sebagai pada persamaan berturut-turut (5.1), (5.2), (5.3), dan (5.4).

c. Meregresikan $\boldsymbol{l}$ pada $\hat{\mathbf{Z}}$ untuk mendapatkan $\hat{\boldsymbol{\delta}}$

Persamaannya adalah $\underset{T\times 1}{\boldsymbol{l}} = \underset{T\times(p+2)}{\hat{\mathbf{Z}}}\underset{(p+2)\times 1}{\boldsymbol{\delta}} + \underset{T\times 1}{\boldsymbol{\varepsilon}}$, dengan mencari

$\dfrac{\partial(\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon})}{\partial(\boldsymbol{\delta})} = 0$ maka didapatkan $\underset{(p+2)\times 1}{\hat{\boldsymbol{\delta}}} = \left(\underset{(p+2)\times T}{\hat{\mathbf{Z}}'}\underset{T\times(p+2)}{\hat{\mathbf{Z}}}\right)^{-1}\underset{(p+2)\times T}{\hat{\mathbf{Z}}'}\underset{T\times 1}{\boldsymbol{l}}$

dengan $\underset{T\times(p+2)}{\hat{\mathbf{Z}}} = \underset{T\times T}{\mathbf{P_H}}\underset{T\times(p+2)}{\mathbf{Z}}$. Vektor $\underset{(p+2)\times 1}{\hat{\boldsymbol{\delta}}}$ diurai menjadi:

$$\underset{(p+2)\times 1}{\hat{\boldsymbol{\delta}}} = \left(\underset{(p+2)\times T}{\mathbf{Z}'}\underset{T\times T}{\mathbf{P}_{\mathbf{H}}'}\underset{T\times T}{\mathbf{P_H}}\underset{T\times(p+2)}{\mathbf{Z}}\right)^{-1}\underset{(p+2)\times T}{\mathbf{Z}'}\underset{T\times T}{\mathbf{P}_{\mathbf{H}}'}\underset{T\times 1}{\boldsymbol{l}}, \qquad (5.6)$$

karena $\underset{T\times T}{\mathbf{P}'_{\mathbf{H}}}\underset{T\times T}{\mathbf{P_H}} = \underset{T\times T}{\mathbf{P_H}}$ (idempoten) dan $\underset{T\times T}{\mathbf{P_H}} = \underset{T\times T}{\mathbf{P}'_{\mathbf{H}}}$ (simetris) maka persamaan (5.6) dapat ditulis kembali menjadi $\underset{(p+2)\times 1}{\hat{\boldsymbol{\delta}}} = \left(\underset{(p+2)\times T}{\mathbf{Z}'}\underset{T\times T}{\mathbf{P_H}}\underset{T\times(p+2)}{\mathbf{Z}}\right)^{-1}\underset{(p+2)\times T}{\mathbf{Z}'}\underset{T\times T}{\mathbf{P_H}}\underset{T\times 1}{\boldsymbol{l}}$. Dengan menggantikan $\underset{T\times(p+2)}{\mathbf{Z}} = \left(\underset{T\times(p+1)}{\boldsymbol{K}} \;\middle|\; \underset{T\times T}{\mathbf{W}}\underset{T\times 1}{\boldsymbol{l}}\right)$ maka dapat dituliskan kembali sebagai berikut: $\underset{(p+2)\times 1}{\hat{\boldsymbol{\delta}}} = \left(\left(\begin{array}{c} \underset{(p+1)\times T}{\boldsymbol{K}'} \\ \hline \underset{1\times T}{(\mathbf{W}\boldsymbol{l})'} \end{array}\right)\underset{T\times T}{\mathbf{P}_H}\left(\underset{T\times(p+1)}{\boldsymbol{K}} \;\middle|\; \underset{T\times 1}{\mathbf{W}\boldsymbol{l}}\right)\right)^{-1}\left(\begin{array}{c} \underset{(p+1)\times T}{\boldsymbol{K}'} \\ \hline \underset{1\times T}{(\mathbf{W}\boldsymbol{l})'} \end{array}\right)\underset{T\times T}{\mathbf{P}_H}\underset{T\times 1}{\boldsymbol{l}}$ atau

$$\underset{(p+2)\times 1}{\hat{\boldsymbol{\delta}}} = \left(\begin{array}{c|c} \underset{(p+1)\times(p+1)}{\boldsymbol{K}'\mathbf{P}_H\boldsymbol{K}} & \underset{(p+1)\times 1}{\boldsymbol{K}'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \\ \hline \underset{1\times(p+1)}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\boldsymbol{K}} & \underset{1\times 1}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \end{array} \right)^{-1} \left(\begin{array}{c} \underset{(p+1)\times T}{\boldsymbol{K}'} \\ \hline \underset{1\times T}{(\mathbf{W}\boldsymbol{l})'} \end{array} \right) \underset{T\times T}{\mathbf{P}_H} \underset{T\times 1}{\boldsymbol{l}}.$$

Selanjutnya akan dicari $\left(\begin{array}{c|c} \underset{(p+1)\times(p+1)}{\boldsymbol{K}'\mathbf{P}_H\boldsymbol{K}} & \underset{(p+1)\times 1}{\boldsymbol{K}'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \\ \hline \underset{1\times(p+1)}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\boldsymbol{K}} & \underset{1\times 1}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \end{array} \right)^{-1}$ dengan memisalkan $\mathbf{B}_1 = \underset{(p+1)\times(p+1)}{\boldsymbol{K}'\mathbf{P}_H\boldsymbol{K}}$; $\mathbf{B}_2 = \underset{(p+1)\times 1}{\boldsymbol{K}'\mathbf{P}_H\mathbf{W}\boldsymbol{l}}$; $\mathbf{B}_3 = \underset{1\times(p+1)}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\boldsymbol{K}}$ dan $\mathbf{B}_4 = \underset{1\times 1}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\mathbf{W}\boldsymbol{l}}$.

Jika $\mathbf{B}_{E1} = \left[\mathbf{B}_1 - \mathbf{B}_2\mathbf{B}_4^{-1}\mathbf{B}_3\right]^{-1}$, maka

$$\underset{(p+1)\times(p+1)}{\mathbf{B}_{E1}} = \left[\underset{(p+1)\times(p+1)}{\boldsymbol{K}'\mathbf{P}_H\boldsymbol{K}} - \left(\underset{(p+1)\times 1}{\boldsymbol{K}'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \right) \left(\underset{1\times 1}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \right)^{-1} \underset{1\times(p+1)}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\boldsymbol{K}} \right]^{-1}. \quad (5.7)$$

Jika $\mathbf{B}_{E2} = \left[-\mathbf{B}_{E1}\mathbf{B}_2\mathbf{B}_4^{-1}\right]$, maka

$$\underset{(p+1)\times 1}{\mathbf{B}_{E2}} = -\left[\underset{(p+1)\times(p+1)}{\boldsymbol{K}'\mathbf{P}_H\boldsymbol{K}} - \left(\underset{(p+1)\times 1}{\boldsymbol{K}'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \right) \left(\underset{1\times 1}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \right)^{-1} \underset{1\times(p+1)}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\boldsymbol{K}} \right]^{-1} \times \underset{(p+1)\times 1}{\boldsymbol{K}'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \left(\underset{1\times 1}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \right)^{-1}. \quad (5.8)$$

$$\mathbf{B}_{E3} = -\left(\underset{1\times 1}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \right)^{-1} \underset{1\times(p+1)}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\boldsymbol{K}} \times \left[\underset{(p+1)\times(p+1)}{\boldsymbol{K}'\mathbf{P}_H\boldsymbol{K}} - \left(\underset{(p+1)\times 1}{\boldsymbol{K}'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \right) \left(\underset{1\times 1}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \right)^{-1} \underset{1\times(p+1)}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\boldsymbol{K}} \right]^{-1}. \quad (5.9)$$

Jika $\mathbf{B}_{E3} = \left[-\mathbf{B}_4^{-1}\mathbf{B}_3\mathbf{B}_{E1}\right]$, maka

Jika $\mathbf{B}_{E4} = \left[\mathbf{B}_4^{-1} - \mathbf{B}_4^{-1}\mathbf{B}_3\mathbf{B}_{E2}\right]$, maka

$$\underset{1\times 1}{\mathbf{B}_{E4}} = \left(\underset{1\times 1}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \right)^{-1} + \left(\underset{1\times 1}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \right)^{-1} \underset{1\times(p+1)}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\boldsymbol{K}} \left(\left(\underset{(p+1)\times(p+1)}{\boldsymbol{K}'\mathbf{P}_H\boldsymbol{K}} - \left(\underset{(p+1)\times 1}{\boldsymbol{K}'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \right) \left(\underset{1\times 1}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \right)^{-1} \underset{1\times(p+1)}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\boldsymbol{K}} \right)^{-1} \underset{(p+1)\times 1}{\boldsymbol{K}'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \left(\underset{1\times 1}{(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\mathbf{W}\boldsymbol{l}} \right)^{-1} \right). \quad (5.10)$$

Selanjutnya $\underset{(p+2)\times 1}{\hat{\boldsymbol{\delta}}}$ diurai kembali menjadi

$$\underset{(p+2)\times 1}{\hat{\boldsymbol{\delta}}} = \left(\begin{array}{c|c} \underset{(p+1)\times(p+1)}{\mathbf{B}_{E1}} & \underset{(p+1)\times 1}{\mathbf{B}_{E2}} \\ \hline \underset{1\times(p+1)}{\mathbf{B}_{E3}} & \underset{1\times 1}{\mathbf{B}_{E4}} \end{array} \right) \left(\begin{array}{c} \underset{(p+1)\times T}{\boldsymbol{K}'} \\ \hline \underset{1\times T}{(\mathbf{W}\boldsymbol{l})'} \end{array} \right) \underset{T\times T}{\mathbf{P}_H} \underset{T\times 1}{\boldsymbol{l}}$$

$$\text{atau} \quad \underset{(p+2)\times 1}{\hat{\boldsymbol{\delta}}} = \left(\begin{array}{c} \underset{(p+1)\times 1}{\mathbf{B}_{E1}\boldsymbol{K}'\mathbf{P}_H\boldsymbol{l} + \mathbf{B}_{E2}(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\boldsymbol{l}} \\ \hline \underset{1\times 1}{\mathbf{B}_{E3}\boldsymbol{K}'\mathbf{P}_H\boldsymbol{l} + \mathbf{B}_{E4}(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\boldsymbol{l}} \end{array} \right), \tag{5.11}$$

sehingga $\underset{(p+1)\times 1}{\hat{\boldsymbol{\beta}}} = \underset{(p+1)\times 1}{\mathbf{B}_{E1}\boldsymbol{K}'\mathbf{P}_H\boldsymbol{l} + \mathbf{B}_{E2}(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\boldsymbol{l}}$ dan $\hat{\lambda} = \underset{1\times 1}{\mathbf{B}_{E3}\boldsymbol{K}'\mathbf{P}_H\boldsymbol{l} + \mathbf{B}_{E4}(\mathbf{W}\boldsymbol{l})'\mathbf{P}_H\boldsymbol{l}}$.

Nilai B_{E1}, B_{E2}, B_{E3}, B_{E4}, dan P_H sesuai dengan persamaan berturut-turut (5.7), (5.8), (5.9), (5.10), dan (5.5).

5.2. Estimasi Parameter Model SERM-SEM

Estimasi parameter model SERM-SEM menggunakan metode *generalized method of moment* (GMM). Model hasil estimasi SAR-SEM diperoleh nilai $\hat{\boldsymbol{l}}_i$, sedangkan selisih antara $\boldsymbol{l}_i$ dan $\hat{\boldsymbol{l}}_i$ merupakan nilai residual yang dinotasikan $\hat{\mathbf{u}}$. Nilai $\hat{\mathbf{u}}$ ini akan digunakan sebagai vektor pengamatan untuk variabel random $\mathbf{u}$ pada model SERM-SEM.

Persamaan *error* model SERM-SEM (4.2) didapatkan nilai $\boldsymbol{\varepsilon} = \mathbf{u} - \rho\mathbf{W}\mathbf{u}$. Misalkan $\bar{\mathbf{u}} = \mathbf{W}\mathbf{u}$ maka *error* dapat ditulis menjadi $\boldsymbol{\varepsilon} = \mathbf{u} - \rho\bar{\mathbf{u}}$

Berikut adalah langkah-langkah manipulasi untuk mendapatkan persamaan momen:

1. Persaamaan momen pertama
   $\boldsymbol{\varepsilon} = \mathbf{u} - \rho\mathbf{W}\mathbf{u}$ dengan memisalkan $\bar{\mathbf{u}} = \mathbf{W}\mathbf{u}$ maka *error* dapat ditulis menjadi
   $\boldsymbol{\varepsilon} = \mathbf{u} - \rho\bar{\mathbf{u}}$, selanjutnya persamaan tersebut dikuadratkan dan menjadi $\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon} = (\mathbf{u} - \rho\bar{\mathbf{u}})'(\mathbf{u} - \rho\bar{\mathbf{u}})$ atau $\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon} = \mathbf{u}'\mathbf{u} - 2\rho\mathbf{u}'\bar{\mathbf{u}} + \rho^2\bar{\mathbf{u}}'\bar{\mathbf{u}}$. Semua elemen dipindah ke sebelah kanan maka persamaan berubah menjadi $2\rho\mathbf{u}'\bar{\mathbf{u}} - \rho^2\bar{\mathbf{u}}'\bar{\mathbf{u}} + \boldsymbol{\varepsilon}'\boldsymbol{\varepsilon} - \mathbf{u}'\mathbf{u} = 0$, selanjutnya dibagi dengan T untuk semua elemen menjadi

$2\rho T^{-1}\mathbf{u}'\bar{\mathbf{u}} - \rho^2 T^{-1}\bar{\mathbf{u}}'\bar{\mathbf{u}} + T^{-1}\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon} - T^{-1}\mathbf{u}'\mathbf{u} = 0$. Sehingga didapatkan persamaan momen sebagai berikut :

$2\rho T^{-1}E(\mathbf{u}'\bar{\mathbf{u}}) - \rho^2 T^{-1}E(\bar{\mathbf{u}}'\bar{\mathbf{u}}) + T^{-1}E(\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}) - T^{-1}E(\mathbf{u}'\mathbf{u}) = 0$.

Karena $E(T^{-1}\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}) = tr(\boldsymbol{\Theta}_\rho)$ maka:

$$2\rho T^{-1}E(\mathbf{u}'\bar{\mathbf{u}}) - \rho^2 T^{-1}E(\bar{\mathbf{u}}'\bar{\mathbf{u}}) + tr(\boldsymbol{\Theta}_\rho) - T^{-1}E(\mathbf{u}'\mathbf{u}) = 0 . \qquad (5.12)$$

2. Persaamaan momen kedua

   $\boldsymbol{\varepsilon} = \mathbf{u} - \rho\mathbf{W}\mathbf{u}$ dengan memisalkan $\bar{\mathbf{u}} = \mathbf{W}\mathbf{u}$ maka *error* dapat ditulis menjadi $\boldsymbol{\varepsilon} = \mathbf{u} - \rho\bar{\mathbf{u}}$. Masing-masing elemen pada persamaan tersebut dikalikan dengan **W** menjadi $\mathbf{W}\boldsymbol{\varepsilon} = \mathbf{W}\mathbf{u} - \rho\mathbf{W}\bar{\mathbf{u}}$. Dengan memisalkan $\bar{\boldsymbol{\varepsilon}} = \mathbf{W}\boldsymbol{\varepsilon}$ dan $\bar{\bar{\mathbf{u}}} = \mathbf{W}\bar{\mathbf{u}}$ maka persamaan berubah menjadi $\bar{\boldsymbol{\varepsilon}} = \bar{\mathbf{u}} - \rho\bar{\bar{\mathbf{u}}}$ yang selanjutnya dikuadratkan menjadi $\bar{\boldsymbol{\varepsilon}}'\bar{\boldsymbol{\varepsilon}} = (\bar{\mathbf{u}} - \rho\bar{\bar{\mathbf{u}}})'(\bar{\mathbf{u}} - \rho\bar{\bar{\mathbf{u}}})$ atau $\bar{\boldsymbol{\varepsilon}}'\bar{\boldsymbol{\varepsilon}} = \bar{\mathbf{u}}'\bar{\mathbf{u}} - 2\rho\bar{\mathbf{u}}'\bar{\bar{\mathbf{u}}} + \rho^2\bar{\bar{\mathbf{u}}}'\bar{\bar{\mathbf{u}}}$. Semua elaman pada persamaan tersebut dipindah ke sebelah kanan dan selanjutnya masing-masing elemen dibagi dengan T, maka persamaan menjadi $2\rho n^{-1}\bar{\mathbf{u}}'\bar{\bar{\mathbf{u}}} - \rho^2 n^{-1}\bar{\bar{\mathbf{u}}}'\bar{\bar{\mathbf{u}}} + n^{-1}\bar{\boldsymbol{\varepsilon}}'\bar{\boldsymbol{\varepsilon}} - n^{-1}\bar{\mathbf{u}}'\bar{\mathbf{u}} = 0$.

   Persamaan momen adalah

   Persamaan momen adalah

   $2\rho T^{-1}E(\bar{\mathbf{u}}'\bar{\bar{\mathbf{u}}}) - \rho^2 T^{-1}E(\bar{\bar{\mathbf{u}}}'\bar{\bar{\mathbf{u}}}) + T^{-1}E(\bar{\boldsymbol{\varepsilon}}'\bar{\boldsymbol{\varepsilon}}) - T^{-1}E(\bar{\mathbf{u}}'\bar{\mathbf{u}}) = 0$, dimana $\bar{\boldsymbol{\varepsilon}} = \mathbf{W}\boldsymbol{\varepsilon}$, $\bar{\boldsymbol{\varepsilon}}'\bar{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}'\mathbf{W}'\mathbf{W}\boldsymbol{\varepsilon}$, dan $E(T^{-1}\bar{\boldsymbol{\varepsilon}}'\bar{\boldsymbol{\varepsilon}}) = tr(\mathbf{W}'\boldsymbol{\Theta}_\rho\mathbf{W}) = tr(\boldsymbol{\Theta}_\rho)tr(\mathbf{W}'\mathbf{W})$

   maka persamaan momen menjadi

   $$2\rho T^{-1}E(\bar{\mathbf{u}}'\bar{\bar{\mathbf{u}}}) - \rho^2 T^{-1}E(\bar{\bar{\mathbf{u}}}'\bar{\bar{\mathbf{u}}}) + tr(\boldsymbol{\Theta}_\rho)tr(\mathbf{W}'\mathbf{W}) - T^{-1}E(\bar{\mathbf{u}}'\bar{\mathbf{u}}) = 0 . \qquad (5.13)$$

3. Persaamaan momen ketiga

   Persamaan $\boldsymbol{\varepsilon} = \mathbf{u} - \rho\bar{\mathbf{u}}$ dan $\bar{\boldsymbol{\varepsilon}} = \bar{\mathbf{u}} - \rho\bar{\bar{\mathbf{u}}}$ dikalikan menjadi $\boldsymbol{\varepsilon}'\bar{\boldsymbol{\varepsilon}} = \mathbf{u}'\bar{\mathbf{u}} - \rho\mathbf{u}'\bar{\bar{\mathbf{u}}} - \rho\bar{\mathbf{u}}'\bar{\mathbf{u}} + \rho^2\bar{\mathbf{u}}'\bar{\bar{\mathbf{u}}}$. Semua eleman pada persamaan tersebut dipindah ke sebelah kanan menjadi $\rho\mathbf{u}'\bar{\bar{\mathbf{u}}} + \rho\bar{\mathbf{u}}'\bar{\mathbf{u}} - \rho^2\bar{\mathbf{u}}'\bar{\bar{\mathbf{u}}} + \boldsymbol{\varepsilon}'\bar{\boldsymbol{\varepsilon}} - \mathbf{u}'\bar{\mathbf{u}} = 0$ atau

   $\rho(\mathbf{u}'\bar{\bar{\mathbf{u}}} + \bar{\mathbf{u}}'\bar{\mathbf{u}}) - \rho^2\bar{\mathbf{u}}'\bar{\bar{\mathbf{u}}} + \boldsymbol{\varepsilon}'\bar{\boldsymbol{\varepsilon}} - \mathbf{u}'\bar{\mathbf{u}} = 0$ dan selanjutnya masing-masing elemen dibagi dengan T menjadi $\rho T^{-1}(\mathbf{u}'\bar{\bar{\mathbf{u}}} + \bar{\mathbf{u}}'\bar{\mathbf{u}}) - \rho^2 T^{-1}\bar{\mathbf{u}}'\bar{\bar{\mathbf{u}}} + n^{-1}\boldsymbol{\varepsilon}'\bar{\boldsymbol{\varepsilon}} - T^{-1}\mathbf{u}'\bar{\mathbf{u}} = 0$.

   Persamaan momennya adalah

$$\rho T^{-1}E\left(\mathbf{u}'\bar{\bar{\mathbf{u}}}+\bar{\mathbf{u}}'\bar{\mathbf{u}}\right)-\rho^{2}T^{-1}E\left(\bar{\mathbf{u}}'\bar{\bar{\mathbf{u}}}\right)+T^{-1}E\left(\boldsymbol{\varepsilon}'\bar{\boldsymbol{\varepsilon}}\right)-T^{-1}E\left(\mathbf{u}'\bar{\mathbf{u}}\right)=0,$$

dimana

$$\begin{aligned}E\left(T^{-1}\boldsymbol{\varepsilon}'\bar{\boldsymbol{\varepsilon}}\right)&=E\left(T^{-1}\boldsymbol{\varepsilon}'tr\left(\mathbf{W}\right)\boldsymbol{\varepsilon}\right)\\&=E\left(T^{-1}\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}\right)tr\left(\mathbf{W}\right)\\&=E\left(T^{-1}\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}\right)0\\&=0,\end{aligned}$$

maka persamaan momen menjadi

$$\rho T^{-1}E\left(\mathbf{u}'\bar{\bar{\mathbf{u}}}+\bar{\mathbf{u}}'\bar{\mathbf{u}}\right)-\rho^{2}T^{-1}E\left(\bar{\mathbf{u}}'\bar{\bar{\mathbf{u}}}\right)-T^{-1}E\left(\mathbf{u}'\bar{\mathbf{u}}\right)=0. \qquad (5.14)$$

Terdapat 3 persamaan momen yaitu persamaan (5.14), (5.14) dan (5.14) sebagai berikut:

$$2\rho T^{-1}E\left(\mathbf{u}'\bar{\mathbf{u}}\right)-\rho^{2}T^{-1}E\left(\bar{\mathbf{u}}'\bar{\mathbf{u}}\right)+tr\left(\boldsymbol{\Theta}_{\rho}\right)-T^{-1}E\left(\mathbf{u}'\mathbf{u}\right)=0$$

$$2\rho T^{-1}E\left(\bar{\mathbf{u}}'\bar{\bar{\mathbf{u}}}\right)-\rho^{2}T^{-1}E\left(\bar{\bar{\mathbf{u}}}'\bar{\bar{\mathbf{u}}}\right)+tr\left(\boldsymbol{\Theta}_{\rho}\right)tr\left(\mathbf{W}'\mathbf{W}\right)-T^{-1}E\left(\bar{\mathbf{u}}'\bar{\mathbf{u}}\right)=0$$

$$\rho T^{-1}E\left(\mathbf{u}'\bar{\bar{\mathbf{u}}}+\bar{\mathbf{u}}'\bar{\mathbf{u}}\right)-\rho^{2}T^{-1}E\left(\bar{\mathbf{u}}'\bar{\bar{\mathbf{u}}}\right)-T^{-1}E\left(\mathbf{u}'\bar{\mathbf{u}}\right)=0$$

Persamaan tersebut dapat disajikan dalam matrik berikut ini :

$$\begin{bmatrix}2T^{-1}E\left(\mathbf{u}'\bar{\mathbf{u}}\right) & -T^{-1}E\left(\bar{\mathbf{u}}'\bar{\mathbf{u}}\right) & 1\\ 2T^{-1}E\left(\bar{\mathbf{u}}'\bar{\bar{\mathbf{u}}}\right) & -T^{-1}E\left(\bar{\bar{\mathbf{u}}}'\bar{\bar{\mathbf{u}}}\right) & Tr\left(\mathbf{W}'\mathbf{W}\right)\\ T^{-1}E\left(\mathbf{u}'\bar{\bar{\mathbf{u}}}+\bar{\mathbf{u}}'\bar{\mathbf{u}}\right) & -T^{-1}E\left(\bar{\mathbf{u}}'\bar{\bar{\mathbf{u}}}\right) & 0\end{bmatrix}\begin{bmatrix}\rho\\ \rho^{2}\\ tr\left(\boldsymbol{\Theta}_{\rho}\right)\end{bmatrix}-\begin{bmatrix}T^{-1}E\left(\mathbf{u}'\mathbf{u}\right)\\ T^{-1}E\left(\bar{\mathbf{u}}'\bar{\mathbf{u}}\right)\\ T^{-1}E\left(\mathbf{u}'\bar{\mathbf{u}}\right)\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

atau diringkas menjadi $\boldsymbol{\Gamma}\boldsymbol{\alpha}-\boldsymbol{\gamma}=\mathbf{0}$

Terdapat satu penaksir yang akan diduga yaitu ρ, sedangkan $\boldsymbol{\Theta}_{\rho}$ mengandung unsur ρ. Penaksir diperoleh dari $\boldsymbol{\Gamma}$ dan $\boldsymbol{\gamma}$. Berdasarkan matrik persamaan momen maka didapatkan bahwa:

$$\boldsymbol{\Gamma}=\begin{bmatrix}2T^{-1}E\left(\mathbf{u}'\bar{\mathbf{u}}\right) & -T^{-1}E\left(\bar{\mathbf{u}}'\bar{\mathbf{u}}\right) & 1\\ 2T^{-1}E\left(\bar{\mathbf{u}}'\bar{\bar{\mathbf{u}}}\right) & -T^{-1}E\left(\bar{\bar{\mathbf{u}}}'\bar{\bar{\mathbf{u}}}\right) & Tr\left(\mathbf{W}'\mathbf{W}\right)\\ T^{-1}E\left(\mathbf{u}'\bar{\bar{\mathbf{u}}}+\bar{\mathbf{u}}'\bar{\mathbf{u}}\right) & -T^{-1}E\left(\bar{\mathbf{u}}'\bar{\bar{\mathbf{u}}}\right) & 0\end{bmatrix},$$

$$\boldsymbol{\Gamma}==\begin{bmatrix}2T^{-1}\hat{\mathbf{u}}'\hat{\bar{\mathbf{u}}} & -T\hat{\bar{\mathbf{u}}}'\hat{\bar{\mathbf{u}}} & 1\\ 2T^{-1}\hat{\bar{\mathbf{u}}}'\hat{\bar{\bar{\mathbf{u}}}} & -T\hat{\bar{\bar{\mathbf{u}}}}'\hat{\bar{\bar{\mathbf{u}}}} & Tr\left(\mathbf{W}'\mathbf{W}\right)\\ T^{-1}\left(\hat{\mathbf{u}}'\hat{\bar{\bar{\mathbf{u}}}}+\hat{\bar{\mathbf{u}}}'\hat{\bar{\mathbf{u}}}\right) & -T\hat{\bar{\mathbf{u}}}'\hat{\bar{\bar{\mathbf{u}}}} & 0\end{bmatrix}$$

$$\boldsymbol{\alpha} = \begin{bmatrix} \rho \\ \rho^2 \\ tr\left(\boldsymbol{\Theta}_\rho\right) \end{bmatrix},$$

$$\boldsymbol{\gamma} = \begin{bmatrix} T^{-1}E\left(\mathbf{u}'\mathbf{u}\right) \\ T^{-1}E\left(\bar{\mathbf{u}}'\bar{\mathbf{u}}\right) \\ T^{-1}E\left(\mathbf{u}'\bar{\mathbf{u}}\right) \end{bmatrix} = \frac{1}{T}\begin{bmatrix} \hat{\mathbf{u}}'\hat{\mathbf{u}} \\ \hat{\bar{\mathbf{u}}}'\hat{\bar{\mathbf{u}}} \\ \hat{\mathbf{u}}'\hat{\bar{\mathbf{u}}} \end{bmatrix}.$$

Maka didapatkan penaksir $\boldsymbol{\Gamma}$ adalah **G**, yaitu :

$$\hat{\boldsymbol{\Gamma}} = \mathbf{G} = \begin{bmatrix} 2T^{-1}\hat{\mathbf{u}}'\mathbf{W}\hat{\mathbf{u}} & -T\hat{\mathbf{u}}'\mathbf{W}'\mathbf{W}\hat{\mathbf{u}} & 1 \\ 2T^{-1}\hat{\mathbf{u}}'\mathbf{W}'\mathbf{W}^2\hat{\mathbf{u}} & -T\hat{\mathbf{u}}'(\mathbf{W}^2)'\mathbf{W}^2\hat{\mathbf{u}} & Tr\left(\mathbf{W}'\mathbf{W}\right) \\ T^{-1}\left(\hat{\mathbf{u}}'\mathbf{W}^2\hat{\mathbf{u}} + \hat{\mathbf{u}}'\mathbf{W}'\mathbf{W}\hat{\mathbf{u}}\right) & -T\hat{\mathbf{u}}'\mathbf{W}'\mathbf{W}^2\hat{\mathbf{u}} & 0 \end{bmatrix} \quad (5.15)$$

.

Sedangkan penaksir $\boldsymbol{\gamma}$ adalah **g**, yaitu :

$$\hat{\gamma} = \mathbf{g} = \frac{1}{T}\begin{bmatrix} \hat{\mathbf{u}}'\hat{\mathbf{u}} \\ \hat{\mathbf{u}}'\mathbf{W}'\mathbf{W}\hat{\mathbf{u}} \\ \hat{\mathbf{u}}'\mathbf{W}\hat{\mathbf{u}} \end{bmatrix}, \quad (5.16)$$

Nilai residual dinotasikan dengan $\hat{\mathbf{u}}$ merupakan selisih antara $\boldsymbol{l}_i$ dan $\hat{\boldsymbol{l}}_i$, sedangkan $\underset{T\times 1}{\boldsymbol{l}} = \underset{T\times(p+2)}{\mathbf{Z}}\underset{(p+2)\times 1}{\boldsymbol{\delta}} + \underset{T\times 1}{\boldsymbol{\varepsilon}}$. Nilai residual dapat dituliskan kembali $\underset{T\times 1}{\hat{\mathbf{u}}} = \underset{T\times 1}{\boldsymbol{l}} - \underset{T\times 1}{\hat{\boldsymbol{l}}}$ atau $\underset{T\times 1}{\hat{\mathbf{u}}} = \underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+2)}{\mathbf{Z}}\underset{(p+2)\times 1}{\hat{\boldsymbol{\delta}}}$. Nilai $\hat{\boldsymbol{\delta}}$ sesuai dengan persamaan (5.11) dan $\underset{T\times(p+2)}{\mathbf{Z}} = \left(\underset{T\times(p+1)}{\boldsymbol{K}} \mid \underset{T\times T}{\mathbf{W}}\underset{T\times 1}{\boldsymbol{l}}\right)$.

Persamaan empiris dari kondisi momen dapat dituliskan kembali sebagai berikut :

$$\underset{3\times 1}{\mathbf{g}} = \underset{3\times 3}{\mathbf{G}}\underset{3\times 1}{\boldsymbol{\alpha}} - \underset{3\times 1}{\mathbf{v}}. \quad (5.17)$$

Penaksiran menggunakan *generalized method of moment* (GMM) didefinisikan sebagai hasil meminimalkan jumlah kuadrat residual atau $\mathbf{v}'\mathbf{v}$. Dari persamaan (5.17) didapatkan nilai $\underset{3\times 1}{\mathbf{v}} = \underset{3\times 3}{\mathbf{G}}\underset{3\times 1}{\boldsymbol{\alpha}} - \underset{3\times 1}{\mathbf{g}}$ sehingga didapatkan nilai kuadrat residualnya adalah

$$\underset{1\times3}{\mathbf{v}'}\underset{3\times1}{\mathbf{v}} = \left(\underset{3\times3}{\mathbf{G}}\underset{3\times1}{\boldsymbol{\alpha}} - \underset{3\times1}{\mathbf{g}}\right)'\left(\underset{3\times3}{\mathbf{G}}\underset{3\times1}{\boldsymbol{\alpha}} - \underset{3\times1}{\mathbf{g}}\right)$$

$$\underset{1\times3}{\mathbf{v}'}\underset{3\times1}{\mathbf{v}} = \left(\underset{1\times3}{\boldsymbol{\alpha}'}\underset{3\times3}{\mathbf{G}'} - \underset{1\times3}{\mathbf{g}'}\right)\left(\underset{3\times3}{\mathbf{G}}\underset{3\times1}{\boldsymbol{\alpha}} - \underset{3\times1}{\mathbf{g}}\right)$$

$$\underset{1\times3}{\mathbf{v}'}\underset{3\times1}{\mathbf{v}} = \underset{1\times3}{\boldsymbol{\alpha}'}\underset{3\times3}{\mathbf{G}'}\underset{3\times3}{\mathbf{G}}\underset{3\times1}{\boldsymbol{\alpha}} - \underset{1\times3}{\boldsymbol{\alpha}'}\underset{3\times3}{\mathbf{G}'}\underset{3\times1}{\mathbf{g}} - \underset{1\times3}{\mathbf{g}'}\underset{3\times3}{\mathbf{G}}\underset{3\times1}{\boldsymbol{\alpha}} + \underset{1\times3}{\mathbf{g}'}\underset{3\times1}{\mathbf{g}}$$

$$\underset{1\times1}{\left(\mathbf{v}'\mathbf{v}\right)} = \underset{1\times1}{\left(\boldsymbol{\alpha}'\mathbf{G}'\mathbf{G}\boldsymbol{\alpha}\right)} - \underset{1\times1}{\left(\boldsymbol{\alpha}'\mathbf{G}'\mathbf{g}\right)} - \underset{1\times1}{\left(\mathbf{g}'\mathbf{G}\boldsymbol{\alpha}\right)} + \underset{1\times1}{\left(\mathbf{g}'\mathbf{g}\right)}$$

$\boldsymbol{\alpha}'\mathbf{G}'\mathbf{g}$ skalar bentuknya simetris, sehingga $\boldsymbol{\alpha}'\mathbf{G}'\mathbf{g} = \left(\boldsymbol{\alpha}'\mathbf{G}'\mathbf{g}\right)' = \mathbf{g}'\mathbf{G}\boldsymbol{\alpha} = \left(\boldsymbol{\alpha}\mathbf{G}'\mathbf{g}\right)'$. Oleh karena itu persamaan nilai kuadrat residual menjadi $\underset{1\times3}{\mathbf{v}'}\underset{3\times1}{\mathbf{v}} = \underset{1\times3}{\boldsymbol{\alpha}'}\underset{3\times3}{\mathbf{G}'}\underset{3\times3}{\mathbf{G}}\underset{3\times1}{\boldsymbol{\alpha}} - 2\left(\underset{3\times3}{\mathbf{G}'}\underset{3\times1}{\mathbf{g}}\right)'\underset{1\times3}{\boldsymbol{\alpha}} + \underset{1\times3}{\mathbf{g}'}\underset{3\times1}{\mathbf{g}}$.

Nilai taksiran $\boldsymbol{\alpha}$ diperoleh dengan cara meminimalkan nilai kuadrat residual, yaitu :

$$\frac{\partial \mathbf{v}'\mathbf{v}}{\partial \boldsymbol{\alpha}} = 0$$

$$2\underset{3\times3}{\mathbf{G}'}\underset{3\times3}{\mathbf{G}}\underset{3\times1}{\boldsymbol{\alpha}} - 2\left(\underset{3\times3}{\mathbf{G}'}\underset{3\times1}{\mathbf{g}}\right) = 0$$

$$\underset{3\times3}{\mathbf{G}'}\underset{3\times3}{\mathbf{G}}\underset{3\times1}{\boldsymbol{\alpha}} = \underset{3\times3}{\mathbf{G}'}\underset{3\times1}{\mathbf{g}}$$

$$\underset{3\times1}{\boldsymbol{\alpha}} = \left(\underset{3\times3}{\mathbf{G}'}\underset{3\times3}{\mathbf{G}}\right)^{-1}\underset{3\times3}{\mathbf{G}'}\underset{3\times1}{\mathbf{g}}$$

sehingga nilai taksiran α adalah $\underset{3\times1}{\hat{\boldsymbol{\alpha}}} = \left(\underset{3\times3}{\mathbf{G}'}\underset{3\times3}{\mathbf{G}}\right)^{-1}\underset{3\times3}{\mathbf{G}'}\underset{3\times1}{\mathbf{g}}$ atau

$$\underset{3\times1}{\begin{bmatrix} \rho \\ \rho^2 \\ tr\left(\boldsymbol{\Theta}_{\rho}\right) \end{bmatrix}} = \left(\underset{3\times3}{\mathbf{G}'}\underset{3\times3}{\mathbf{G}}\right)^{-1}\underset{3\times3}{\mathbf{G}'}\underset{3\times1}{\mathbf{g}}$$

dengan nilai matrik $\mathbf{G}$ sesuai dengan persamaan (5.15) dan nilai matrik $\mathbf{g}$ sesuai dengan persamaan (5.16).

BAB 6 | UJI DEPENDENSI SPASIAL

Ada beberapa uji spasial dependensi pada model regresi spasial standar. Diantaranya uji *Moran's I*, uji *Lagrange Multiplier* untuk dependensi spasial *error* dan uji *Lagrange Multiplier* untuk dependensi spasial *autoregressive*. Uji *Moran's I* untuk autokorelasi *error* spasial merupakan uji secara umum, sedangkan uji *Lagrange Multiplier* bersifat lebih spesifik. Pada buku ini menggunakan uji *Lagrange Multiplier* untuk menguji spasial dependensi.

6.1. Uji Dependensi Spasial Model SAR-SEM

Uji dependensi spasial pada model SAR-SEM (tanpa asumsi distribusi *error* model sama dengan model SAR tradisional) menggunakan uji *Langrange Multiplier* (LM) akan diurai pada Teorema 4.

Teorema 4

Jika model SAR-SEM sebagaimana pada persamaan (2.16) dengan distribusi *error* sebagaimana persamaan (4.1) maka didapatkan uji *Langrange Multiplier* adalah $\underset{1\times 1}{\mathrm{LM}_\lambda} = \dfrac{-\left(\underset{1\times 1}{p}\left(\underset{T\times T}{\mathbf{W}}\ \underset{T\times(p+1)}{\boldsymbol{K}}\ \underset{(p+1)\times 1}{\hat{\boldsymbol{\beta}}}\right)'\underset{T\times 1}{\tilde{\boldsymbol{\varepsilon}}}\right)^2}{\underset{1\times 1}{p}\ \underset{1\times 1}{D}}$ dan di bawah H_0, statistik uji LM_λ mengikuti distribusi $\chi^2_{(1)}$ dimana nilai

p adalah $\underset{1\times 1}{p} = \left(\underset{1\times B}{\mathbf{\Lambda}_{\mathbf{y}}^{'}} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\mathbf{\Lambda}_{\mathbf{y}}} \right)$ dan nilai D adalah

$\underset{1\times 1}{D} = \left(\underset{T\times 1}{\mathbf{e}} \underset{1\times 1}{\hat{\boldsymbol{\eta}}_t} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\hat{\boldsymbol{\beta}}} \right)^{'} \underset{T\times T}{\mathbf{W}} \underset{T\times T}{\mathbf{W}^{'}} \left(\underset{T\times 1}{\mathbf{e}} \underset{1\times 1}{\hat{\boldsymbol{\eta}}_t} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\hat{\boldsymbol{\beta}}} \right).$

Bukti Teorema 4

Model SAR-SEM sebagaimana pada persamaan (2.16) mempunyai nilai *error* adalah $\underset{T\times 1}{\boldsymbol{\varepsilon}} = \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}} \right) \underset{T\times T}{\mathbf{I}} - \underset{T\times(p+1)}{\mathbf{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}}$. Misalkan $\underset{T\times T}{\mathbf{A}} = \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}} \right)$ sehingga dapat ditulis kembali $\underset{T\times T}{\mathbf{A}} \underset{T\times 1}{\boldsymbol{l}} = \underset{T\times(p+1)}{\mathbf{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}} + \underset{T\times 1}{\boldsymbol{\varepsilon}}$ dan persamaan *error* adalah

$$\underset{T\times 1}{\boldsymbol{\varepsilon}} = \underset{T\times T}{\mathbf{A}} \underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\mathbf{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}}. \quad (6.1)$$

Uji *Langrange Multiplier* (LM) merupakan uji berdasarkan estimasi di bawah hipotesis nol. *Jacobian* untuk persamaan *error* (6.1) adalah $J = \left| \frac{\partial \boldsymbol{\varepsilon}}{\partial \boldsymbol{l}} \right| = |\mathbf{I} - \lambda \mathbf{W}| = |\mathbf{A}|$. Dengan fungsi *Gaussian* dapat diperoleh fungsi *likelihood* untuk *error* sebagai berikut: $L(\sigma^2, \boldsymbol{\varepsilon}) = c(\boldsymbol{\varepsilon}) |V|^{-1/2} \exp\left[-\frac{1}{2} \boldsymbol{\varepsilon}^{'} \mathbf{V}^{-1} \boldsymbol{\varepsilon} \right]$ dimana **V** adalah matrik *variance-covariance* dari $\boldsymbol{\varepsilon}$, yaitu $\boldsymbol{\Theta} = \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}} \right) \left(\underset{1\times B}{\mathbf{\Lambda}_y^{'}} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\mathbf{\Lambda}_y} \right)^{-1} \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}} \right)^{'}$ dengan memisalkan $\underset{1\times 1}{p} = \left(\underset{1\times B}{\mathbf{\Lambda}_y^{'}} \underset{B\times B}{\mathbf{\Theta}_{\varepsilon^*}^{-1}} \underset{B\times 1}{\mathbf{\Lambda}_y} \right)$ dan $\underset{T\times T}{\mathbf{A}} = \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}} \right)$ maka $\underset{T\times T}{\boldsymbol{\Theta}} = \underset{T\times T}{\mathbf{A}} \underset{1\times 1}{p^{-1}} \underset{T\times T}{\mathbf{A}^{'}} = \underset{1\times 1}{p^{-1}} \underset{T\times T}{\mathbf{A}} \underset{T\times T}{\mathbf{A}^{'}}$.

Fungsi *likelihood* $\boldsymbol{l}$ pada model SAR-SEM diperoleh dengan mensubsitusikan $\boldsymbol{\varepsilon}$ dan mengalikan dengan *Jacobian*, sehingga didapatkan fungsi *likelihood* untuk model SAR-SEM adalah:

$$L(\lambda, \boldsymbol{\beta}, \boldsymbol{\Theta}; \boldsymbol{l}) = |\boldsymbol{\Theta}|^{-1/2} |\mathbf{A}| \exp\left[-\frac{1}{2} (\mathbf{A}\boldsymbol{l} - \boldsymbol{K}\boldsymbol{\beta})^{'} \boldsymbol{\Theta}^{-1} (\mathbf{A}\boldsymbol{l} - \boldsymbol{K}\boldsymbol{\beta}) \right].$$

Sedangkan fungsi *ln likelihood* untuk model SAR-SEM adalah:

$\mathcal{L}(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l}) = -\frac{1}{2}\ln|\boldsymbol{\Theta}| + \ln|\mathbf{A}| - \frac{1}{2}(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon})$, dengan $\underset{T\times T}{\boldsymbol{\Theta}} = \underset{1\times 1}{p}^{-1}\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{A}'}$,

$\underset{1\times 1}{p} = \left(\underset{1\times B}{\boldsymbol{\Lambda}_y'}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\boldsymbol{\Lambda}_y}\right)$, $\underset{T\times T}{\mathbf{A}} = \left(\underset{T\times T}{\mathbf{I}} - \lambda\underset{T\times T}{\mathbf{W}}\right)$, dan $\underset{T\times 1}{\boldsymbol{\varepsilon}} = \underset{T\times T}{\mathbf{A}}\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}$.

Proses mendapatkan turunan pertama, turunan kedua, elemen matrik informasi, dan elemen baris dan kolom pertama dari invers matrik informasi dilampirkan pada Lampiran 5. Berikut ini adalah hasil turunan pertama fungsi *ln likelihood* untuk $L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})$ terhadap λ dan terhadap $\boldsymbol{\beta}$:

a. Turunan pertama fungsi *ln likelihood* terhadap λ adalah

$$\underset{(p+1)\times 1}{\frac{\partial\mathcal{L}(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}}} = \underset{1\times 1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)'\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)$$

b. Turunan pertama fungsi *ln likelihood* terhadap $\boldsymbol{\beta}$ adalah

$$\underset{(p+1)\times 1}{\frac{\partial\mathcal{L}(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}}} = \underset{1\times 1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\boldsymbol{K}}\right)'\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right).$$

Hasil turunan kedua fungsi *ln likelihood* untuk $L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})$ terhadap λ dan terhadap $\boldsymbol{\beta}$:

a. Turunan kedua $\frac{\partial\mathcal{L}(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\lambda}$ terhadap λ adalah

$$\underset{1\times 1}{\frac{\partial^2\mathcal{L}(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial^2\lambda}} = \underset{1\times 1}{p}\left(\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)'\left(\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\underset{T\times T}{\mathbf{A}^{-1}}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\right)\times$$
$$\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right) - \left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\times$$
$$\left(\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\left(\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right)$$

b. Turunan kedua $\frac{\partial\mathcal{L}(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}}$ terhadap $\boldsymbol{\beta}$:

$$\underset{(p+1)\times(p+1)}{\frac{\partial\mathcal{L}(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'}} = \underset{1\times 1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\boldsymbol{K}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\boldsymbol{K}}.$$

c. Turunan kedua $\frac{\partial\mathcal{L}(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\lambda}$ terhadap $\boldsymbol{\beta}$ adalah

$$\frac{\partial^2 \mathcal{L}(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\lambda\partial\boldsymbol{\beta}'}_{1\times(p+1)} = \underset{1\times1}{p}\left(\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\boldsymbol{K}}\right)-\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\boldsymbol{K}}\right)\right)$$

d. Turunan kedua $\frac{\partial \mathcal{L}(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}}$ terhadap λ

$$\frac{\partial^2 \mathcal{L}(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}\partial\lambda'}_{(p+1)\times1} = \underset{1\times1}{p}\left(\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\boldsymbol{K}}\right)'\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)-\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\boldsymbol{K}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right)$$

Breusch dan Pagan (1980) mendefinisikan statistik uji LM sebagai berikut: $\text{LM} = \hat{\mathbf{D}}'\hat{\boldsymbol{\psi}}^{-1}\hat{\mathbf{D}}$, dimana $\hat{\boldsymbol{\psi}}^{-1}$ merupakan elemen dari invers matriks informasi $\tilde{\boldsymbol{\psi}}_\theta$ berukuran kxk yang elemen-elemen di dalamnya merupakan turunan kedua terhadap masing-masing parameter yang diestimasi: $\tilde{\boldsymbol{\psi}}_\theta = E\left[\frac{\partial^2 L(\theta)}{\partial\theta\partial\theta'}\right]$. Matrik informasi untuk model SAR-SEM pada saat $\lambda = 0$ adalah $\tilde{\boldsymbol{\psi}}_\theta = \begin{pmatrix} \underset{1\times1}{\tilde{\psi}_{\lambda\lambda}} & \underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\lambda\boldsymbol{\beta}}} \\ \underset{(p+1)\times1}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\lambda}} & \underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}'}} \end{pmatrix}$.

Berikut adalah elemen matrik informasi pada saat $\lambda = 0$:

a. Elemen matrik (1,1)

$$\underset{1\times1}{\tilde{\psi}_{\lambda\lambda}} = \underset{1\times1}{p}\left(\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)-2\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{W}'}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right).$$

b. Elemen matrik (2,2) $\underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\beta\beta}} = \underset{1\times1}{p}\left(\underset{T\times(p+1)}{\boldsymbol{K}'}\underset{T\times(p+1)}{\boldsymbol{K}}\right)$.

c. Elemen matrik (1,2)

$$\underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\lambda\beta}} = \underset{1\times1}{p}\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times(p+1)}{\boldsymbol{K}} - \left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}\right)\right].$$

d. Elemen matrik (2,1)

$$\underset{(p+1)\times1}{\tilde{\boldsymbol{\psi}}_{\beta\lambda}} = \underset{1\times1}{p}\left[\underset{T\times(p+1)}{\boldsymbol{K}}\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right) - \left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}\right)\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right].$$

Elemen invers matrik informasi (1,1) yaitu $\underset{1\times1}{\tilde{\psi}_{\lambda\lambda}} = \left(\underset{1\times1}{\tilde{\psi}_{\lambda\lambda}} - \underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\lambda\beta}}\left(\underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\beta\beta}}\right)^{-1}\underset{(p+1)\times1}{\tilde{\boldsymbol{\psi}}_{\beta\lambda}}\right)^{-1}$ dan jika disederhanakan menjadi

$$\underset{1\times1}{\tilde{\psi}_{\lambda\lambda}^{-1}} = -\underset{1\times1}{p^{-1}}\left[\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\hat{\boldsymbol{\eta}}_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right)'\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{W}'}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\hat{\boldsymbol{\eta}}_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right)\right]^{-1}.$$

Selanjutnya, nilai statistik uji LM untuk model SAR-SEM adalah: $\mathrm{LM}_\lambda = \hat{\mathbf{D}}_\lambda'\hat{\boldsymbol{\psi}}_{\lambda\lambda}^{-1}\hat{\mathbf{D}}_\lambda$. Karena pengujian dilakukan di bawah H_0, dimana $\lambda=0$ maka diperoleh nilai untuk $\hat{\mathbf{D}}_\lambda$ yang merupakan turunan pertama fungsi *ln likelihood* terhadap λ dengan $\lambda=0$, sebagai berikut:

$$\underset{1\times1}{\hat{D}_\lambda} = \underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right)'\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right).$$

Karena $(\boldsymbol{l}-\boldsymbol{K}\boldsymbol{\beta})$ merupakan *error* dari model regresi OLS maka $\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right) = \underset{T\times1}{\tilde{\boldsymbol{\varepsilon}}}$ sehingga $\underset{1\times1}{\hat{D}_\lambda} = \underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right)'\underset{T\times1}{\tilde{\boldsymbol{\varepsilon}}}$. Nilai statistik uji LM menjadi

$$\mathrm{LM}_\lambda = \left(\underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right)'\underset{T\times1}{\tilde{\boldsymbol{\varepsilon}}}\right)'\left(-\underset{1\times1}{p}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\hat{\boldsymbol{\eta}}_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right)'\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{W}'}\right.$$
$$\left.\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\hat{\boldsymbol{\eta}}_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right)\right)^{-1}\left(\underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right)'\underset{T\times1}{\tilde{\boldsymbol{\varepsilon}}}\right) \quad \text{atau}$$

$$\text{LM}_{\lambda} = \frac{-\left(\underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{W}}\ \underset{T\times(p+1)}{\boldsymbol{K}}\ \underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right)'\underset{T\times1}{\tilde{\boldsymbol{\varepsilon}}}\right)^2}{\underset{1\times1}{p}\left(\underset{T\times1}{\mathbf{e}}\ \underset{1\times1}{\hat{\boldsymbol{\eta}}_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\ \underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right)'\underset{T\times T}{\mathbf{W}}\ \underset{T\times T}{\mathbf{W}'}\left(\underset{T\times1}{\mathbf{e}}\ \underset{1\times1}{\hat{\boldsymbol{\eta}}_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\ \underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right)}$$, dimana nilai

$\underset{1\times1}{p} = \left(\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}'}\ \underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon*}^{-1}}\ \underset{B\times1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)$ dan $\left(\underset{T\times1}{\mathbf{e}}\ \underset{1\times1}{\hat{\boldsymbol{\eta}}_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\ \underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right)'\underset{T\times T}{\mathbf{W}}\ \underset{T\times T}{\mathbf{W}'}\left(\underset{T\times1}{\mathbf{e}}\ \underset{1\times1}{\hat{\boldsymbol{\eta}}_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\ \underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right) = \underset{1\times1}{D}$

maka nilai statistik uji LM menjadi

$$\underset{1\times1}{\text{LM}_{\lambda}} = \frac{-\left(\underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{W}}\ \underset{T\times(p+1)}{\boldsymbol{K}}\ \underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right)'\underset{T\times1}{\tilde{\boldsymbol{\varepsilon}}}\right)^2}{\underset{1\times1}{p}\ \underset{1\times1}{D}}. \tag{6.2}$$

Statistik uji *Langrange Multiplier* didefinisikan oleh Breusch dan Pagan (1980), yaitu $\text{LM} = \tilde{\mathbf{D}}'\tilde{\boldsymbol{\psi}}^{-1}\tilde{\mathbf{D}}$. Jika H0: $\left[\lambda, \rho, \boldsymbol{\alpha}'\right] = 0$ dimana terdapat sejumlah *p* parameter dari $\boldsymbol{\alpha}$ yang berhubungan dengan nonkonstrain, maka menurut Anselin (1988b) $\text{LM} = \tilde{\mathbf{D}}'\tilde{\boldsymbol{\psi}}^{-1}\tilde{\mathbf{D}}$ berdistribusi $\chi^2_{(2+p)}$. Oleh karena itu model SAR-SEM dengan H_0 :

$\lambda = 0$ maka $\text{LM}_{\lambda} \sim \chi^2_{(1)}$ atau $\dfrac{-\left(p\left(\mathbf{W}\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'\tilde{\boldsymbol{\varepsilon}}\right)^2}{pD} \sim \chi^2_{(1)}$.

Daerah penolakan H_0 adalah tolak H_0 jika $\text{LM}_{\lambda} \geq c \Leftrightarrow \dfrac{-\left(p\left(\mathbf{W}\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'\tilde{\boldsymbol{\varepsilon}}\right)^2}{pD} \geq c$. Pengujian di bawah H_0 dan distribusi dari $\dfrac{-\left(p\left(\mathbf{W}\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'\tilde{\boldsymbol{\varepsilon}}\right)^2}{pD}$ adalah $\chi^2_{(1)}$ maka $\alpha = P\left\{\dfrac{-\left(p\left(\mathbf{W}\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'\tilde{\boldsymbol{\varepsilon}}\right)^2}{pD} \geq c\right\}$

dengan nilai c yang bersesuaian, yaitu $\chi^2_{(1;1-\alpha)}$. Oleh karena itu uji

Langrange Multiplier untuk model SAR-SEM (LM_{λ}) menolak H_0 jika

$$\frac{-\left(p\left(\mathbf{W}\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)^{'}\tilde{\boldsymbol{\varepsilon}}\right)^{2}}{pD} \geq \chi^{2}_{(1;1-\alpha)}.$$

6.2. Uji Depensi Spasial Model SERM-SEM

Uji dependensi spasial pada model SERM-SEM (tanpa asumsi distribusi *error* model sama dengan model SAR tradisional) menggunakan uji *Langrange Multiplier* (LM) akan diurai pada Teorema 5.

Teorema 5

Jika model SERM-SEM sebagaimana pada persamaan (2.17) dengan distribusi *error* sebagaimana persamaan (4.2) maka didapatkan uji *Langrange Multiplier* adalah $\mathrm{LM}_{\rho} = \dfrac{\underset{1\times1}{p}\left(\underset{1\times T}{\tilde{\boldsymbol{\varepsilon}}^{'}}\ \underset{T\times T}{\mathbf{W}}\ \underset{T\times1}{\tilde{\boldsymbol{\varepsilon}}}\right)^{2}}{\underset{1\times1}{D}}$ dan di bawah H_0, statistik uji LM_{ρ} mengikuti distribusi $\chi^{2}_{(1)}$ dimana nilai p adalah $\underset{1\times1}{p} = \left(\underset{1\times B}{\boldsymbol{\Lambda}_{y}^{'}}\ \underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^{*}}^{-1}}\ \underset{B\times1}{\boldsymbol{\Lambda}_{y}}\right)$ dan nilai D adalah

$$\left(\underset{T\times1}{\mathbf{e}}\ \underset{1\times1}{\hat{\eta}_{t}} - \underset{T\times(p+1)}{\boldsymbol{K}}\ \underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right)^{'}\underset{T\times T}{\mathbf{W}}\ \underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\mathbf{e}}\ \underset{1\times1}{\hat{\eta}_{t}} - \underset{T\times(p+1)}{\boldsymbol{K}}\ \underset{(p+1)\times1}{\hat{\boldsymbol{\beta}}}\right) = \underset{1\times1}{D}.$$

Bukti Teorema 5

Model SERM-SEM sebagaimana pada persamaan (2.17) dapat ditulis kembali, yaitu $\left(\underset{T\times T}{\mathbf{I}} - \rho\underset{T\times T}{\mathbf{W}}\right)\underset{T\times1}{\boldsymbol{l}} = \left(\underset{T\times T}{\mathbf{I}} - \rho\underset{T\times T}{\mathbf{W}}\right)\underset{T\times(p+1)}{\boldsymbol{K}}\ \underset{(p+1)\times1}{\boldsymbol{\beta}} + \underset{T\times1}{\boldsymbol{\varepsilon}}$, sehingga nilai *error* adalah $\underset{T\times1}{\boldsymbol{\varepsilon}} = \left(\underset{T\times T}{\mathbf{I}} - \rho\underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\ \underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$. Misalkan $\underset{T\times T}{\mathbf{A}} = \left(\underset{T\times T}{\mathbf{I}} - \rho\underset{T\times T}{\mathbf{W}}\right)$ sehingga dapat ditulis kembali $\underset{T\times T}{\mathbf{A}}\ \underset{T\times1}{\boldsymbol{l}} = \underset{T\times T}{\mathbf{A}}\ \underset{T\times(p+1)}{\boldsymbol{K}}\ \underset{(p+1)\times1}{\boldsymbol{\beta}} + \underset{T\times1}{\boldsymbol{\varepsilon}}$ dan persamaan *error* adalah

$$\underset{T\times1}{\boldsymbol{\varepsilon}} = \underset{T\times T}{\mathbf{A}}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right). \tag{6.3}$$

Uji *Langrange Multiplier* (LM) merupakan uji berdasarkan estimasi di bawah hipotesis nol. *Jacobian* untuk persamaan *error* (6.3) adalah $J = \left|\frac{\partial\boldsymbol{\varepsilon}}{\partial\boldsymbol{l}}\right| = |\mathbf{I} - \rho\mathbf{W}| = |\mathbf{A}|$. Dengan fungsi *Gaussian* dapat diperoleh fungsi *likelihood* untuk *error* sebagai berikut: $L(\sigma^2, \boldsymbol{\varepsilon}) = c(\boldsymbol{\varepsilon})|V|^{-1/2}\exp\left[-\frac{1}{2}\boldsymbol{\varepsilon}'\mathbf{V}^{-1}\boldsymbol{\varepsilon}\right]$ dimana $\mathbf{V}$ adalah matrik *variance-covariance* dari $\boldsymbol{\varepsilon}$, yaitu $\boldsymbol{\Theta}_\rho = \left(\underset{T\times T}{\mathbf{I}} - \rho\underset{T\times T}{\mathbf{W}}\right)\left(\underset{1\times B}{\boldsymbol{\Lambda}'_{\mathbf{y}}}\underset{B\times B}{\boldsymbol{\Theta}^{-1}_{\boldsymbol{\varepsilon}^*}}\underset{B\times1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1}\underset{T\times T}{\mathbf{I}}\left(\underset{T\times T}{\mathbf{I}} - \rho\underset{T\times T}{\mathbf{W}}\right)'$ dengan memisalkan $\underset{1\times1}{p} = \left(\underset{1\times B}{\boldsymbol{\Lambda}'_y}\underset{B\times B}{\boldsymbol{\Theta}^{-1}_{\varepsilon^*}}\underset{B\times1}{\boldsymbol{\Lambda}_y}\right)$ dan $\underset{T\times T}{\mathbf{A}} = \left(\underset{T\times T}{\mathbf{I}} - \rho\underset{T\times T}{\mathbf{W}}\right)$ maka $\underset{T\times T}{\boldsymbol{\Theta}} = \underset{T\times T}{\mathbf{A}}\underset{1\times1}{p^{-1}}\underset{T\times T}{\mathbf{I}}\underset{T\times T}{\mathbf{A}'} = \underset{1\times1}{p^{-1}}\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{I}}\underset{T\times T}{\mathbf{A}'} = \underset{1\times1}{p^{-1}}\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{A}'}$.

Fungsi *likelihood* $\boldsymbol{l}$ pada model SERM-SEM diperoleh dengan mensubsitusikan $\boldsymbol{\varepsilon}$ dan mengalikan dengan *Jacobian*, sehingga didapatkan fungsi *likelihood* untuk model SERM-SEM adalah:

$$L(\rho, \boldsymbol{\beta}, \boldsymbol{\Theta}; \boldsymbol{l}) = |\boldsymbol{\Theta}|^{-1/2}|\mathbf{A}|\exp\left[-\frac{1}{2}(\mathbf{A}\boldsymbol{l} - \mathbf{A}\boldsymbol{K}\boldsymbol{\beta})'\boldsymbol{\Theta}^{-1}(\mathbf{A}\boldsymbol{l} - \mathbf{A}\boldsymbol{K}\boldsymbol{\beta})\right]$$

$$L(\rho, \boldsymbol{\beta}, \boldsymbol{\Theta}; \boldsymbol{l}) = |\boldsymbol{\Theta}|^{-1/2}|\mathbf{A}|\exp\left[-\frac{1}{2}(\mathbf{A}\boldsymbol{l} - \mathbf{A}\boldsymbol{K}\boldsymbol{\beta})'\left(p^{-1}\mathbf{A}\mathbf{A}'\right)^{-1}(\mathbf{A}\boldsymbol{l} - \mathbf{A}\boldsymbol{K}\boldsymbol{\beta})\right]$$

$$L(\rho, \boldsymbol{\beta}, \boldsymbol{\Theta}; \boldsymbol{l}) = |\boldsymbol{\Theta}|^{-1/2}|\mathbf{A}|\exp\left[-\frac{1}{2}p(\mathbf{A}\boldsymbol{l} - \mathbf{A}\boldsymbol{K}\boldsymbol{\beta})'\left(\mathbf{A}\mathbf{A}'\right)^{-1}(\mathbf{A}\boldsymbol{l} - \mathbf{A}\boldsymbol{K}\boldsymbol{\beta})\right]$$

Sedangkan fungsi *ln likelihood* untuk model SERM-SEM adalah:

$$\mathcal{L}(\rho, \boldsymbol{\beta}, \boldsymbol{\Theta}; \boldsymbol{l}) = -\frac{1}{2}\ln|\boldsymbol{\Theta}| + \ln|\mathbf{A}| - \frac{1}{2}p(\mathbf{A}\boldsymbol{l} - \mathbf{A}\boldsymbol{K}\boldsymbol{\beta})'\left(\mathbf{A}\mathbf{A}'\right)^{-1}(\mathbf{A}\boldsymbol{l} - \mathbf{A}\boldsymbol{K}\boldsymbol{\beta}),$$

dengan $\underset{T\times T}{\boldsymbol{\Theta}} = \underset{1\times1}{p^{-1}}\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{A}'}$, $\underset{1\times1}{p} = \left(\underset{1\times B}{\boldsymbol{\Lambda}'_y}\underset{B\times B}{\boldsymbol{\Theta}^{-1}_{\varepsilon^*}}\underset{B\times1}{\boldsymbol{\Lambda}_y}\right)$, $\underset{T\times T}{\mathbf{A}} = \left(\underset{T\times T}{\mathbf{I}} - \rho\underset{T\times T}{\mathbf{W}}\right)$.

Proses mendapatkan turunan pertama, turunan kedua, elemen matrik informasi, dan elemen baris dan kolom pertama dari invers matrik informasi dilampirkan pada Lampiran 6. Berikut ini adalah hasil turunan pertama fungsi *ln likelihood* untuk $L(\rho, \boldsymbol{\beta}, \boldsymbol{\Theta}; \boldsymbol{l})$ terhadap ρ dan terhadap $\boldsymbol{\beta}$:

a. Turunan pertama fungsi *ln likelihood* terhadap ρ adalah

$$\underset{1\times 1}{\frac{\partial \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \rho}} = \underset{1\times 1}{p}\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)' \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right).$$

b. Turunan pertama fungsi *ln likelihood* terhadap $\boldsymbol{\beta}$ adalah

$$\underset{(p+1)\times 1}{\frac{\partial \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \boldsymbol{\beta}}} = \underset{1\times 1}{p}\underset{(p+1)\times T}{\boldsymbol{K}'}\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right).$$

Hasil turunan kedua fungsi *ln likelihood* untuk $L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})$ terhadap ρ dan terhadap :

a. Turunan kedua terhadap ρ atau $\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \rho^2}$ adalah

$$\underset{1\times 1}{\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \rho^2}} = -\underset{1\times 1}{p}\left[\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)' \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right]$$

b. Turunan kedua terhadap $\boldsymbol{\beta}$ atau $\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}'}$:

$$\underset{(p+1)\times(p+1)}{\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}'}} = -\underset{1\times 1}{p}\underset{(p+1)\times T}{\boldsymbol{K}'}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times(p+1)}{\mathbf{I}} = -\underset{1\times 1}{p}\underset{(p+1)\times T}{\boldsymbol{K}'}\underset{T\times(p+1)}{\boldsymbol{K}}.$$

c. Turunan kedua $\frac{\partial \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \rho}$ terhadap $\boldsymbol{\beta}$ adalah

$$\underset{1\times(p+1)}{\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \rho\partial \boldsymbol{\beta}'}} = -2\underset{1\times 1}{p}\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)' \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}$$

d. Turunan kedua $\frac{\partial \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \boldsymbol{\beta}}$ terhadap ρ

$$\underset{(p\times 1)\times 1}{\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \boldsymbol{\beta}\partial \rho}} = \mathbf{0}$$

Breusch dan Pagan (1980) mendefinisikan statistik uji LM sebagai berikut: $\text{LM} = \hat{\mathbf{D}}'\hat{\boldsymbol{\psi}}^{-1}\hat{\mathbf{D}}$, dimana $\hat{\boldsymbol{\psi}}^{-1}$ merupakan elemen dari invers matriks informasi $\tilde{\boldsymbol{\psi}}_{\theta}$ berukuran $k\times k$ yang elemen-elemen di dalamnya merupakan turunan kedua terhadap masing-masing parameter yang diestimasi: $\tilde{\boldsymbol{\psi}}_{\theta} = E\left[\frac{\partial^2 L(\theta)}{\partial\theta\partial\theta'}\right]$. Matrik informasi untuk model SERM-SEM pada saat $\rho = 0$ adalah $\tilde{\boldsymbol{\psi}}_{\theta} = \begin{pmatrix} \underset{1\times1}{\tilde{\boldsymbol{\psi}}_{\rho\rho}} & \underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\rho\boldsymbol{\beta}}} \\ \underset{(p+1)\times1}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\rho}} & \underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}'}} \end{pmatrix}$.

Berikut adalah elemen matrik informasi pada saat $\rho = 0$:

a. Elemen matrik (1,1):

$$\underset{1\times1}{\tilde{\boldsymbol{\psi}}_{\rho\rho}} = \underset{1\times1}{p}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\eta_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\eta_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

b. Elemen matrik (2,2): $\underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}}} = \underset{1\times1}{p}\left(\underset{(p+1)\times T}{\boldsymbol{K}'}\underset{T\times(p+1)}{\boldsymbol{K}}\right)$.

c. Elemen matrik (1,2): $\underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\rho\boldsymbol{\beta}}} = 2\underset{1\times1}{p}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\eta_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}$.

d. Elemen matrik (2,1): $\underset{(p\times1)\times1}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\rho}} = \mathbf{0}$.

Elemen invers matrik informasi (1,1) yaitu $\underset{1\times1}{\tilde{\psi}^{-1}_{\rho\rho}} = \left(\underset{1\times1}{\tilde{\psi}_{\rho\rho}} - \underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\rho\beta}}\left(\underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\beta\beta}}\right)^{-1}\underset{(p+1)\times1}{\tilde{\boldsymbol{\psi}}_{\beta\rho}}\right)^{-1}$ dan jika disederhanakan menjadi

$$\underset{1\times1}{\tilde{\boldsymbol{\psi}}^{-1}_{\rho\rho}} = \underset{1\times1}{p^{-1}}\left[\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\eta_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\eta_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right]^{-1}.$$

Selanjutnya, nilai statistik uji LM untuk model SERM-SEM adalah: $\mathrm{LM}_{\rho} = \hat{\mathbf{D}}'_{\rho}\hat{\boldsymbol{\psi}}^{-1}_{\rho\rho}\hat{\mathbf{D}}_{\rho}$. Karena pengujian dilakukan di bawah H_0, dimana $\rho = 0$ maka diperoleh nilai untuk $\hat{\mathbf{D}}_{\rho}$ yang merupakan turunan pertama fungsi *ln likelihood* terhadap ρ dengan $\rho = 0$, sebagai berikut:

$$\underset{1\times1}{\hat{D}_{\rho}} = \underset{1\times1}{p}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

Karena $(\boldsymbol{l}-\boldsymbol{K}\boldsymbol{\beta})$ merupakan *error* dari model regresi OLS maka $\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\hat{\boldsymbol{\beta}}}\right) = \underset{T\times 1}{\tilde{\boldsymbol{\varepsilon}}}$ sehingga $\underset{1\times 1}{\hat{D}_{\rho}} = \underset{1\times 1}{p}\, \underset{1\times T}{\tilde{\boldsymbol{\varepsilon}}'} \underset{T\times T}{\mathbf{W}} \underset{T\times 1}{\tilde{\boldsymbol{\varepsilon}}}$. Nilai statistik uji LM menjadi

$$\mathrm{LM}_{\rho} = \frac{\left(\underset{1\times 1}{p}\, \underset{1\times T}{\tilde{\boldsymbol{\varepsilon}}'} \underset{T\times T}{\mathbf{W}} \underset{T\times 1}{\tilde{\boldsymbol{\varepsilon}}}\right)'\left(\underset{1\times 1}{p}\, \underset{1\times T}{\tilde{\boldsymbol{\varepsilon}}'} \underset{T\times T}{\mathbf{W}} \underset{T\times 1}{\tilde{\boldsymbol{\varepsilon}}}\right)}{\underset{1\times 1}{p}\left(\underset{T\times 1}{\mathbf{e}} \underset{1\times 1}{\hat{\eta}_t} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\hat{\boldsymbol{\beta}}}\right)' \underset{T\times T}{\mathbf{W}} \underset{T\times T}{\mathbf{W}} \left(\underset{T\times 1}{\mathbf{e}} \underset{1\times 1}{\hat{\eta}_t} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\hat{\boldsymbol{\beta}}}\right)}$$

$$\mathrm{LM}_{\rho} = \frac{\underset{1\times 1}{p}\left(\underset{1\times T}{\tilde{\boldsymbol{\varepsilon}}'} \underset{T\times T}{\mathbf{W}} \underset{T\times 1}{\tilde{\boldsymbol{\varepsilon}}}\right)'\left(\underset{1\times T}{\tilde{\boldsymbol{\varepsilon}}'} \underset{T\times T}{\mathbf{W}} \underset{T\times 1}{\tilde{\boldsymbol{\varepsilon}}}\right)}{\left(\underset{T\times 1}{\mathbf{e}} \underset{1\times 1}{\hat{\eta}_t} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\hat{\boldsymbol{\beta}}}\right)' \underset{T\times T}{\mathbf{W}} \underset{T\times T}{\mathbf{W}} \left(\underset{T\times 1}{\mathbf{e}} \underset{1\times 1}{\hat{\eta}_t} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\hat{\boldsymbol{\beta}}}\right)}$$

$$\mathrm{LM}_{\rho} = \frac{\underset{1\times 1}{p}\left(\underset{1\times T}{\tilde{\boldsymbol{\varepsilon}}'} \underset{T\times T}{\mathbf{W}} \underset{T\times 1}{\tilde{\boldsymbol{\varepsilon}}}\right)^2}{\left(\underset{T\times 1}{\mathbf{e}} \underset{1\times 1}{\hat{\eta}_t} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\hat{\boldsymbol{\beta}}}\right)' \underset{T\times T}{\mathbf{W}} \underset{T\times T}{\mathbf{W}} \left(\underset{T\times 1}{\mathbf{e}} \underset{1\times 1}{\hat{\eta}_t} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\hat{\boldsymbol{\beta}}}\right)}$$

Dengan memisalkan $\left(\underset{T\times 1}{\mathbf{e}} \underset{1\times 1}{\hat{\eta}_t} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\hat{\boldsymbol{\beta}}}\right)' \underset{T\times T}{\mathbf{W}} \underset{T\times T}{\mathbf{W}} \left(\underset{T\times 1}{\mathbf{e}} \underset{1\times 1}{\hat{\eta}_t} - \underset{T\times(p+1)}{\boldsymbol{K}} \underset{(p+1)\times 1}{\hat{\boldsymbol{\beta}}}\right) = \underset{1\times 1}{D}$ maka nilai statistik uji LM menjadi

$$\mathrm{LM}_{\rho} = \frac{\underset{1\times 1}{p}\left(\underset{1\times T}{\tilde{\boldsymbol{\varepsilon}}'} \underset{T\times T}{\mathbf{W}} \underset{T\times 1}{\tilde{\boldsymbol{\varepsilon}}}\right)^2}{\underset{1\times 1}{D}}. \tag{6.4}$$

Statistik uji *Langrange Multiplier* didefinisikan oleh Breusch dan Pagan (1980), yaitu $\mathrm{LM} = \tilde{\mathbf{D}}'\tilde{\boldsymbol{\psi}}^{-1}\tilde{\mathbf{D}}$. Jika H0: $\left[\lambda, \rho, \boldsymbol{\alpha}'\right] = 0$ dimana terdapat sejumlah p parameter dari $\boldsymbol{\alpha}$ yang berhubungan dengan nonkonstrain, maka menurut Anselin (1988b) $\mathrm{LM} = \tilde{\mathbf{D}}'\tilde{\boldsymbol{\psi}}^{-1}\tilde{\mathbf{D}}$ berdistribusi $\chi^2_{(2+p)}$. Oleh karena itu model SERM-SEM dengan *H0:* $\rho = 0$ maka $\mathrm{LM}_{\rho} \sim \chi^2_{(1)}$ atau $\frac{p\left(\tilde{\boldsymbol{\varepsilon}}'\mathbf{W}\tilde{\boldsymbol{\varepsilon}}\right)^2}{D} \sim \chi^2_{(1)}$

Daerah penolakan H_0 adalah tolak H_0 jika $\text{LM}_\rho \geq c \Leftrightarrow \frac{p\left(\tilde{\boldsymbol{\varepsilon}}'\mathbf{W}\tilde{\boldsymbol{\varepsilon}}\right)^2}{D} \geq c$.

Pengujian di bawah H_0 dan distribusi dari $\frac{p\left(\tilde{\boldsymbol{\varepsilon}}'\mathbf{W}\tilde{\boldsymbol{\varepsilon}}\right)^2}{D}$ adalah $\chi^2_{(1)}$

maka $\alpha = P\left\{\frac{p\left(\tilde{\boldsymbol{\varepsilon}}'\mathbf{W}\tilde{\boldsymbol{\varepsilon}}\right)^2}{D} \geq c\right\}$ dengan nilai c yang bersesuaian, yaitu

$\chi^2_{(1;1-\alpha)}$. Oleh karena itu uji *Langrange Multiplier* untuk model SERM-

SEM ($\mathbf{LM}_\rho$) menolak H_0 jika $\frac{p\left(\tilde{\boldsymbol{\varepsilon}}'\mathbf{W}\tilde{\boldsymbol{\varepsilon}}\right)^2}{D} \geq \chi^2_{(1;1-\alpha)}$.

BAB 7

PENGUJIAN PARAMETER SECARA SIMULTAN MENGGUNAKAN *MAXIMUM LIKELIHOOD RATIO TEST* (MLRT)

7.1. Model SAR-SEM

Uji simultan parameter model SAR-SEM sebagaimana pada persamaan (2.16) dengan distribusi *error* sebagaimana persamaan (4.1) dilakukan untuk mengetahui signifikansi parameter pada model secara simultan dan akan diurai pada Teorema 6.

Teorema 6

Jika model SAR-SEM sebagaimana pada persamaan (2.16) dengan distribusi *error* sebagaimana persamaan (4.1) dan distribusi $\boldsymbol{l}$ sebagaimana pada persamaan (3.8) maka statistik uji MLRT adalah $\left(\Lambda^{-\frac{2}{n}}-1\right)\frac{(n-p)}{(n-1)p}$ dimana $\Lambda^{-\frac{2}{n}}=\left|\frac{\sum_{i=1}^{n}(\boldsymbol{l}_i-\mathbf{e}\eta)'(\boldsymbol{l}_i-\mathbf{e}\eta)}{\sum_{i=1}^{n}\left(\boldsymbol{l}_i-(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}})\right)'\left(\boldsymbol{l}_i-(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}})\right)}\right|$.

Pada tingkat signifikansi α hipotesis nol ditolak jika $\left(\Lambda^{-\frac{2}{n}}-1\right)\frac{(n-p)}{(n-1)p}>F_{(p,n-p(\alpha))}$.

Bukti Teorema 6

Pengujian dilakukan dengan hipotesis sebagai berikut:

H_0 : $\beta_1=\beta_2=\cdots=\beta_p=0$

H_1 : paling sedikit ada satu $\beta_j\neq 0$.

a. Menentukan himpunan parameter di bawah populasi

Model $\underset{T\times 1}{\boldsymbol{l}}=\underset{T\times 1}{\boldsymbol{\beta}_0}+\underset{T\times p}{\boldsymbol{K}}\underset{p\times 1}{\boldsymbol{\beta}}+\lambda\underset{T\times T}{\mathbf{W}}\underset{T\times 1}{\boldsymbol{l}}+\underset{T\times 1}{\boldsymbol{\varepsilon}}$, dimana distribusi

$$\underset{T\times 1}{\boldsymbol{l}}\sim N_T\left(\underset{T\times 1}{\mathbf{e}}\underset{1\times 1}{\eta_t},\left(\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}'}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1}\underset{T\times T}{\mathbf{I}_T}\right)$$

Himpunan parameter di bawah populasi adalah

$\Omega = \{\beta_0, \beta_j, \lambda; j = 1,2,...,p, |\lambda| < 1\} = \{\boldsymbol{\beta}, \lambda\}$

Fungsi kepadatan probabilitas dari $\boldsymbol{l}$ adalah

$$f(\boldsymbol{l}) = \left|\left(\boldsymbol{\Lambda}_y^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\boldsymbol{l} - \mathbf{e}\eta)^{'}\left(\boldsymbol{\Lambda}_y^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)\mathbf{I}_T(\boldsymbol{l} - \mathbf{e}\eta)\right\}$$

Menentukan himpunan parameter di bawah H_0

Di bawah H_0: $\beta_1 = \beta_2 = \cdots = \beta_p = 0$ sehingga $\boldsymbol{l}_0 = \boldsymbol{\beta}_0 + \lambda\mathbf{W}\boldsymbol{l} + \boldsymbol{\varepsilon}$ atau $\boldsymbol{l}_0 = \boldsymbol{l} - \boldsymbol{K}\boldsymbol{\beta}$. Nilai ekspektasi dan varians dari $\boldsymbol{l}_0$ adalah $E(\boldsymbol{l}_0) = \mathbf{e}\eta - \boldsymbol{K}\boldsymbol{\beta}$ dan $\text{var}(\boldsymbol{l}_0) = \left(\boldsymbol{\Lambda}_y^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T$. Distribusi $\boldsymbol{l}_0 \sim N_T\left(\mathbf{e}\eta - \boldsymbol{K}\boldsymbol{\beta}, \left(\boldsymbol{\Lambda}_y^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T\right)$.

Himpunan parameter di bawah H_o adalah $\omega = \{\beta_0, \lambda; |\lambda| < 1\}$

Fungsi kepadatan probabilitas dari $\boldsymbol{l}_0$ adalah:

$$f(\boldsymbol{l}_0) = \left|\left(\boldsymbol{\Lambda}_y^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T\right|^{-\frac{1}{2}} \times$$
$$\exp\left\{-\frac{1}{2}\left(\boldsymbol{l}_0 - (\mathbf{e}\eta - \boldsymbol{K}\boldsymbol{\beta})\right)^{'}\left(\boldsymbol{\Lambda}_y^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)\mathbf{I}_T\left(\boldsymbol{l}_0 - (\mathbf{e}\eta - \boldsymbol{K}\boldsymbol{\beta})\right)\right\}$$

b. Mengambil n sampel random

Ambil n sampel random $\boldsymbol{l}_i$, $i = 1,2,3,\ldots,n$ sehingga didapatkan distribusi $\boldsymbol{l}_i \sim N_T\left(\mathbf{e}\eta_i, \left(\boldsymbol{\Lambda}_y^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T\right)$ dan fungsi *likelihood* di bawah populasi yaitu

$$L(\Omega) = \prod_{i=1}^{n} f(l_i)$$

$$L(\Omega) = \prod_{i=1}^{n}(2\pi)^{-\frac{T}{2}}\left|\left(\boldsymbol{\Lambda}_y^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T\right|^{-\frac{1}{2}} \exp\left\{\frac{1}{2}(\boldsymbol{l}_i - \mathbf{e}\eta)^{'}\left(\boldsymbol{\Lambda}_y^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T(\boldsymbol{l}_i - \mathbf{e}\eta)\right\}$$

$$L(\Omega) = (2\pi)^{-\frac{nT}{2}}\left|\left(\boldsymbol{\Lambda}_y^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T\right|^{-\frac{n}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{l}_i - \mathbf{e}\eta)^{'}\left(\boldsymbol{\Lambda}_y^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)\mathbf{I}_T(\boldsymbol{l}_i - \mathbf{e}\eta)\right\}$$

di bawah H_0 $\boldsymbol{l}_0 \sim N_T\left(\mathbf{e}\eta - \boldsymbol{K}\boldsymbol{\beta}, \left(\boldsymbol{\Lambda}_y^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T\right)$ dan

$\boldsymbol{l}_i \sim N_T\left(\mathbf{e}\eta - \boldsymbol{K}\boldsymbol{\beta}, \left(\boldsymbol{\Lambda}_y^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T\right)$

$$L(\omega) = \prod_{i=1}^{n} f\left(l_{0_i}\right)$$

$$L(\omega) = \prod_{i=1}^{n} (2\pi)^{-\frac{T}{2}} \left| \left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'} \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1} \mathbf{I}_T \right|^{-\frac{1}{2}} \times$$

$$\exp\left\{ \frac{1}{2} \left(\boldsymbol{l}_{0_i} - (\mathbf{e}\eta + \boldsymbol{K}\boldsymbol{\beta})\right)^{'} \left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'} \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1} \mathbf{I}_T \left(\boldsymbol{l}_{0_i} - (\mathbf{e}\eta + \boldsymbol{K}\boldsymbol{\beta})\right) \right\}$$

$$L(\omega) = (2\pi)^{-\frac{nT}{2}} \left| \left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'} \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1} \mathbf{I}_T \right|^{-\frac{n}{2}} \times$$

$$\exp\left\{ -\frac{1}{2} \sum_{i=1}^{n} \left(\boldsymbol{l}_{0_i} - (\mathbf{e}\eta - \boldsymbol{K}\boldsymbol{\beta})\right)^{'} \left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'} \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_{\mathbf{y}}\right) \mathbf{I}_T \left(\boldsymbol{l}_{0_i} - (\mathbf{e}\eta - \boldsymbol{K}\boldsymbol{\beta})\right) \right\}.$$

c. Menentukan *odds ratio*

Di bawah populasi nilai $L(\hat{\Omega})$ adalah

$$L(\hat{\Omega}) = \frac{\exp\left\{ -\frac{1}{2} \sum_{i=1}^{n} (\boldsymbol{l}_i - \mathbf{e}\eta)^{'} \left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'} \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_{\mathbf{y}}\right) \mathbf{I}_T (\boldsymbol{l}_i - \mathbf{e}\eta) \right\}}{(2\pi)^{-\frac{nT}{2}} \left| \left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'} \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1} \mathbf{I}_T \right|^{-\frac{n}{2}}}$$

$$L(\hat{\Omega}) = \frac{\exp\left\{ -\frac{1}{2} tr\left(\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'} \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_{\mathbf{y}}\right) \mathbf{I}_T \sum_{i=1}^{n} (\boldsymbol{l}_i - \mathbf{e}\eta)^{'} (\boldsymbol{l}_i - \mathbf{e}\eta)^{'} \right) \right\}}{(2\pi)^{\frac{nT}{2}} \left| \left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'} \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1} \mathbf{I}_T \right|^{\frac{n}{2}}}$$

Karena $(\boldsymbol{l}_i - \mathbf{e}\eta)^{'} (\boldsymbol{l}_i - \mathbf{e}\eta)^{'} = \left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'} \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1} \mathbf{I}_T$ maka

$\sum_{i=1}^{n} (\boldsymbol{l}_i - \mathbf{e}\eta)^{'} (\boldsymbol{l}_i - \mathbf{e}\eta)^{'} = n\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'} \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1} \mathbf{I}_T$ sehingga

$$L(\hat{\Omega}) = \frac{\exp\left\{ -\frac{1}{2} tr\left(\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'} \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_{\mathbf{y}}\right) \mathbf{I}_T n \left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'} \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1} \mathbf{I}_T \right) \right\}}{(2\pi)^{\frac{nT}{2}} \left| n^{-1} \sum_{i=1}^{n} (\boldsymbol{l}_i - \mathbf{e}\eta)^{'} (\boldsymbol{l}_i - \mathbf{e}\eta)^{'} \right|^{\frac{n}{2}}}$$

$$L(\hat{\Omega}) = \frac{\exp\left\{ -\frac{1}{2} tr(n\mathbf{I}_T) \right\}}{(2\pi)^{\frac{nT}{2}} \left| n^{-1} \sum_{i=1}^{n} (\boldsymbol{l}_i - \mathbf{e}\eta)^{'} (\boldsymbol{l}_i - \mathbf{e}\eta)^{'} \right|^{\frac{n}{2}}}$$

$$L\left(\hat{\Omega}\right)=\frac{\exp\left\{-\frac{1}{2}nT\right\}}{\left(2\pi\right)^{\frac{nT}{2}}\left|n^{-1}\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\mathbf{e}\eta_i\right)^{'}\left(\boldsymbol{l}_i-\mathbf{e}\eta_i\right)^{'}\right|^{\frac{n}{2}}}$$

Di bawah H_0 nilai $L\left(\hat{\omega}\right)$ adalah

$$L\left(\hat{\omega}\right)=\frac{\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)\mathbf{I}_T\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right\}}{\left(2\pi\right)^{\frac{nT}{2}}\left|\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right|^{\frac{n}{2}}}$$

$$L\left(\hat{\omega}\right)=\frac{\exp\left\{-\frac{1}{2}tr\left(\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)\mathbf{I}_T\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right)\right\}}{\left(2\pi\right)^{\frac{nT}{2}}\left|\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right|^{\frac{n}{2}}}$$

Karena nilai $\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)=\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T$

maka $\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)=n\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T$

sehingga

$$L\left(\hat{\omega}\right)=\frac{\exp\left\{-\frac{1}{2}tr\left(\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)\mathbf{I}_T\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right)\right\}}{\left(2\pi\right)^{\frac{nT}{2}}\left|\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right|^{\frac{n}{2}}}$$

$$L\left(\hat{\omega}\right)=\frac{\exp\left\{-\frac{1}{2}tr\left(\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)\mathbf{I}_T n\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right)\right\}}{\left(2\pi\right)^{\frac{nT}{2}}\left|n^{-1}\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right|^{\frac{n}{2}}}$$

$$L\left(\hat{\omega}\right)=\frac{\exp\left\{-\frac{1}{2}tr\left(n\mathbf{I}_T\right)\right\}}{\left(2\pi\right)^{\frac{nT}{2}}\left|n^{-1}\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right|^{\frac{n}{2}}}$$

$$L(\hat{\omega}) = \frac{\exp\left\{-\frac{1}{2}nT\right\}}{(2\pi)^{\frac{nT}{2}} \left| n^{-1}\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right|^{\frac{n}{2}}}$$

d. Nilai *odds ratio* adalah

$$\Lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})}$$

$$\Lambda = \frac{\left| n^{-1}\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_{0_i} - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right|^{\frac{n}{2}}}{\left| n^{-1}\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \mathbf{e}\eta\right)'\left(\boldsymbol{l}_i - \mathbf{e}\eta\right)'\right|^{\frac{n}{2}}}$$

$$\Lambda^{-\frac{2}{n}} = \left| \frac{\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \mathbf{e}\eta\right)'\left(\boldsymbol{l}_i - \mathbf{e}\eta\right)'}{\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)} \right|$$

Jika terdapat matrik $\mathbf{C} = \left(\begin{array}{c|c} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \hline \mathbf{C}_{21} & \mathbf{C}_{22} \end{array}\right)$ maka

$$|\mathbf{C}| = |\mathbf{C}_{11}|\left|\mathbf{C}_{22} - \mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{C}_{12}\right| = |\mathbf{C}_{22}|\left|\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\right|$$

Misalkan terdapat matrik A dengan

$$|\mathbf{C}| = \left| \begin{array}{c|c} \sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right) & \sqrt{n}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)' \\ \hline \sqrt{n}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right) & -1 \end{array} \right|,$$ maka nilai

$|\mathbf{C}_{22}|\left|\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\right|$ adalah

$$= |-1|\left|\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right) - \sqrt{n}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'(-1)^{-1}\sqrt{n}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right|$$

$$= -\left|\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right) + n\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right|$$

Bentuk lain dari persamaan $\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \mathbf{e}\eta\right)'\left(\boldsymbol{l}_i - \mathbf{e}\eta\right)$ adalah sebagai berikut:

$$= \sum_{i=1}^{n} (\boldsymbol{l}_i - \mathbf{e}\eta)' (\boldsymbol{l}_i - \mathbf{e}\eta)$$

$$= \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} + \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)' \left(\boldsymbol{l}_i - \mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} + \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)$$

$$= \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right) + \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right) + \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)$$

$$= \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right) - \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)' \boldsymbol{K}\hat{\boldsymbol{\beta}} + \sum_{i=1}^{n} \left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)' \left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)$$

$$= \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right) + n\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)' \left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)$$

$$\sum_{i=1}^{n} \left(\boldsymbol{l}_i - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)' \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right) = \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)' \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)$$

Karena $\sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)' = 0$ maka

$$\sum_{i=1}^{n} (\boldsymbol{l}_i - \mathbf{e}\eta)' (\boldsymbol{l}_i - \mathbf{e}\eta) = \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right) + n\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)' \left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)$$

Sehingga $|\mathbf{C}_{22}|\left|\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\right| = -\left|\sum_{i=1}^{n} (\boldsymbol{l}_i - \mathbf{e}\eta)' (\boldsymbol{l}_i - \mathbf{e}\eta)\right|$.

Selanjutnya dicari nilai $|\mathbf{C}_{11}|\left|\mathbf{C}_{22} - \mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{C}_{12}\right|$, yaitu

$$= \left|\sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right| \times$$

$$\left|-1 - \sqrt{n}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\left(\sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right)^{-1} \sqrt{n}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'\right|$$

$$= -\left|\sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right| \times$$

$$\left|1 + n\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\left(\sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right)^{-1} \left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'\right|$$

dimana $\sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right) = n\left(\left(\boldsymbol{\Lambda}_y' \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_y\right)^{-1} \mathbf{I}_T\right)$,

sehingga

$$= -\left|\sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right| \left|1 + n\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\left(n\left(\left(\boldsymbol{\Lambda}_y' \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_y\right)^{-1} \mathbf{I}_T\right)\right)^{-1} \left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'\right|$$

$$= -\left|\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right|\left|1+\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\left(\left(\boldsymbol{\Lambda}_y'\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T\right)^{-1}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'\right|$$

Oleh karena itu nilai dari $|\mathbf{C}_{11}||\mathbf{C}_{22}-\mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{C}_{12}|$ adalah

$$|\mathbf{C}_{11}||\mathbf{C}_{22}-\mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{C}_{12}| = -\left|\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right|\times$$
$$\left|1+\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\left(\left(\boldsymbol{\Lambda}_y'\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T\right)^{-1}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'\right|$$

Selanjutnya menyamakan

$$|\mathbf{C}_{11}||\mathbf{C}_{22}-\mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{C}_{12}| = |\mathbf{C}_{22}||\mathbf{C}_{11}-\mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}|$$

$$-\left|\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\mathbf{e}\eta\right)'\left(\boldsymbol{l}_i-\mathbf{e}\eta\right)\right| = -\left|\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right|\times$$
$$\left|1+\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\left(\left(\boldsymbol{\Lambda}_y'\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T\right)^{-1}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'\right|$$

$$\left|\frac{\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\mathbf{e}\eta\right)'\left(\boldsymbol{l}_i-\mathbf{e}\eta\right)}{\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)}\right| = \left|1+\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\left(\left(\boldsymbol{\Lambda}_y'\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T\right)^{-1}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'\right|. \quad (7.1)$$

Diketahui bahwa $\boldsymbol{l}_0 \sim N_T\left(\mathbf{e}\eta-\boldsymbol{K}\boldsymbol{\beta},\left(\boldsymbol{\Lambda}_y'\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T\right)$. Jika $\boldsymbol{l}^* = \boldsymbol{l}_0-\mathbf{e}\eta$ maka $\boldsymbol{E}\left(\boldsymbol{l}^*\right) = E\left(\boldsymbol{l}_0-\mathbf{e}\eta\right) = E\left(\boldsymbol{l}_0\right)-\mathbf{e}\eta = \mathbf{e}\eta-\boldsymbol{K}\boldsymbol{\beta}-\mathbf{e}\eta = -\boldsymbol{K}\boldsymbol{\beta}$

$\operatorname{var}\left(\boldsymbol{l}^*\right) = \operatorname{var}\left(\boldsymbol{l}_0-\mathbf{e}\eta\right) = \operatorname{var}\left(\boldsymbol{l}_0\right) = \left(\boldsymbol{\Lambda}_y'\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathrm{I}_T$. Oleh karena itu, didapatkan distribusi dari $\boldsymbol{l}^*$, yaitu $\boldsymbol{l}^* \sim N\left(-\boldsymbol{K}\boldsymbol{\beta},\left(\boldsymbol{\Lambda}_y'\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T\right)$. Di bawah H_0 : $\beta_1=\beta_2=\cdots=\beta_p=\mathbf{0}$, didapatkan $\left(-\boldsymbol{K}\hat{\boldsymbol{\beta}}-\mathbf{0}\right)'\left(\boldsymbol{\Lambda}_y'\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T\left(-\boldsymbol{K}\hat{\boldsymbol{\beta}}-\mathbf{0}\right) = T^2$ atau $\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'\left(\boldsymbol{\Lambda}_y'\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_y\right)^{-1}\mathbf{I}_T\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right) = \boldsymbol{T}^2$ maka persamaan (7.1) menjadi

$$\left|\frac{\sum_{i=1}^{n}(\boldsymbol{l}_i-\mathbf{e}\eta)'(\boldsymbol{l}_i-\mathbf{e}\eta)}{\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)}\right|=\left|1+T^2\right|$$

$$\Lambda^{-\frac{2}{n}}=1+T^2$$

$$T^2=\Lambda^{-\frac{2}{n}}-1$$

Sedangkan distribusi T^2 *Hotelling* adalah $T^2\sim\frac{(n-1)p}{(n-p)}F_{p,n-p}$.

Daerah penolakan H_0 adalah

$$\alpha=P\left(T^2>\frac{(n-1)p}{(n-p)}F_{(p,n-p(\alpha))}\right)$$

$$\alpha=P\left(\Lambda^{-\frac{2}{n}}-1>\frac{(n-1)p}{(n-p)}F_{(p,n-p(\alpha))}\right)$$

$$\alpha=P\left(\left(\Lambda^{-\frac{2}{n}}-1\right)\frac{(n-p)}{(n-1)p}>F_{(p,n-p(\alpha))}\right)$$

Uji *maximum likelihood ratio* untuk model SAR-SEM berdasarkan uji F dengan statitik uji $\left(\Lambda^{-\frac{2}{n}}-1\right)\frac{(n-p)}{(n-1)p}$ dimana

$$\Lambda^{-\frac{2}{n}}=\left|\frac{\sum_{i=1}^{n}(\boldsymbol{l}_i-\mathbf{e}\eta)'(\boldsymbol{l}_i-\mathbf{e}\eta)}{\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)}\right|$$

Pada tingkat signifikansi α hipotesis nol ditolak jika $\left(\Lambda^{-\frac{2}{n}}-1\right)\frac{(n-p)}{(n-1)p}>F_{(p,n-p(\alpha))}$.

7.2. Model SERM-SEM

Uji simultan parameter model SERM-SEM sebagaimana pada persamaan (2.17) dengan distribusi *error* sebagaimana persamaan (4.2) dilakukan untuk mengetahui signifikansi parameter pada model secara simultan dan akan diurai pada Teorema 7.

Teorema 7

Jika model SERM-SEM sebagaimana pada persamaan (2.17) dengan distribusi *error* sebagaimana persamaan (4.2) dan distribusi $\boldsymbol{l}$ sebagaimana pada persamaan (3.12) maka statistik uji MLRT adalah

$$\left(\Lambda^{-\frac{2}{n}}-1\right)\frac{(n-p)}{(n-1)p} \quad \text{dimana} \quad \Lambda^{-\frac{2}{n}}=\left|\frac{\sum_{i=1}^{n}(\boldsymbol{l}_i-\mathbf{e}\eta)'(\boldsymbol{l}_i-\mathbf{e}\eta)}{\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)}\right|.$$

Pada tingkat signifikansi α hipotesis nol ditolak jika $\left(\Lambda^{-\frac{2}{n}}-1\right)\frac{(n-p)}{(n-1)p}>F_{(p,n-p(\alpha))}$.

Bukti Teorema 7

Pengujian dilakukan dengan hipotesis sebagai berikut:

H_0 : $\beta_1=\beta_2=\cdots=\beta_p=0$

H_1 : paling sedikit ada satu $\beta_j\neq 0$.

a. Menentukan himpunan parameter di bawah populasi

Model $\underset{T\times 1}{\boldsymbol{l}}=\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}+\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{W}}\right)^{-1}\underset{T\times 1}{\boldsymbol{\varepsilon}}$ atau

$\underset{T\times 1}{\boldsymbol{l}}=\underset{T\times 1}{\boldsymbol{\beta}_0}+\underset{T\times p}{\boldsymbol{K}}\underset{p\times 1}{\boldsymbol{\beta}}+\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{W}}\right)^{-1}\underset{T\times 1}{\boldsymbol{\varepsilon}}$, dimana distribusi

$$\underset{T\times 1}{\boldsymbol{l}}\sim N_T\left(\underset{T\times 1}{\mathbf{e}}\underset{1\times 1}{\eta_t},\left(\underset{1\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}'}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1}\underset{T\times T}{\mathbf{I}_T}\right)$$

Himpunan parameter di bawah populasi adalah

$$\Omega=\{\beta_0,\beta_j,\rho;\ j=1,2,\ldots,p,|\rho|<1\}=\{\boldsymbol{\beta},\rho\}$$

Fungsi kepadatan probabilitas dari $\boldsymbol{l}$ adalah

$$f(\boldsymbol{l})=\left|\left(\boldsymbol{\Lambda}_{\mathbf{y}}'\boldsymbol{\Theta}_{\varepsilon^*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}(\boldsymbol{l}-\mathbf{e}\eta)'\left(\boldsymbol{\Lambda}_{\mathbf{y}}'\boldsymbol{\Theta}_{\varepsilon^*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)\mathbf{I}_T(\boldsymbol{l}-\mathbf{e}\eta)\right\}$$

Menentukan himpunan parameter di bawah H_0

Di bawah H_0 : $\beta_1=\beta_2=\cdots=\beta_p=0$ sehingga $\boldsymbol{l}=\boldsymbol{\beta}_0+(\mathbf{I}-\rho\mathbf{W})^{-1}\boldsymbol{\varepsilon}$ atau $\boldsymbol{l}_0=\boldsymbol{l}-\boldsymbol{K}\boldsymbol{\beta}$. Nilai ekspektasi dan varians dari $\boldsymbol{l}_0$ adalah

$E(\boldsymbol{l}_0) = \mathbf{e}\eta - \boldsymbol{K}\boldsymbol{\beta}$ dan $\text{var}(\boldsymbol{l}_0) = \left(\boldsymbol{\Lambda}_\mathbf{y}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_\mathbf{y}\right)^{-1}\mathbf{I}_T$. Distribusi $\boldsymbol{l}_0 \sim N_T\left(\mathbf{e}\eta - \boldsymbol{K}\boldsymbol{\beta}, \left(\boldsymbol{\Lambda}_\mathbf{y}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_\mathbf{y}\right)^{-1}\mathbf{I}_T\right)$.

Himpunan parameter di bawah H_0 adalah $\omega = \{\beta_0, \rho; |\rho| < 1\}$

Fungsi kepadatan probabilitas dari $\boldsymbol{l}_0$ adalah:

$$f(\boldsymbol{l}_0) = \left|\left(\boldsymbol{\Lambda}_\mathbf{y}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_\mathbf{y}\right)^{-1}\mathbf{I}_T\right|^{-\frac{1}{2}} \times$$

$$\exp\left\{-\frac{1}{2}\left(\boldsymbol{l}_0 - (\mathbf{e}\eta - \boldsymbol{K}\boldsymbol{\beta})\right)^{'}\left(\boldsymbol{\Lambda}_\mathbf{y}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_\mathbf{y}\right)\mathbf{I}_T\left(\boldsymbol{l}_0 - (\mathbf{e}\eta - \boldsymbol{K}\boldsymbol{\beta})\right)\right\}$$

b. Mengambil *n* sampel random

Ambil *n* sampel random $\boldsymbol{l}_i$, $i = 1,2,3,\ldots,n$ sehingga didapatkan distribusi $\boldsymbol{l}_i \sim N_T\left(\mathbf{e}\eta_i, \left(\boldsymbol{\Lambda}_\mathbf{y}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_\mathbf{y}\right)^{-1}\mathbf{I}_T\right)$ dan fungsi *likelihood* di bawah populasi yaitu

$$L(\Omega) = \prod_{i=1}^{n} f(\boldsymbol{l}_i)$$

$$L(\Omega) = \prod_{i=1}^{n}(2\pi)^{-\frac{T}{2}}\left|\left(\boldsymbol{\Lambda}_\mathbf{y}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_\mathbf{y}\right)^{-1}\mathbf{I}_T\right|^{-\frac{1}{2}}\exp\left\{\frac{1}{2}(\boldsymbol{l}_i - \mathbf{e}\eta)^{'}\left(\boldsymbol{\Lambda}_\mathbf{y}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_\mathbf{y}\right)\mathbf{I}_T(\boldsymbol{l}_i - \mathbf{e}\eta)\right\}$$

$$L(\Omega) = (2\pi)^{-\frac{nT}{2}}\left|\left(\boldsymbol{\Lambda}_\mathbf{y}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_\mathbf{y}\right)^{-1}\mathbf{I}_T\right|^{-\frac{n}{2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{l}_i - \mathbf{e}\eta)^{'}\left(\boldsymbol{\Lambda}_\mathbf{y}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_\mathbf{y}\right)\mathbf{I}_T(\boldsymbol{l}_i - \mathbf{e}\eta)\right\}$$

di bawah H_0, distribusi $\boldsymbol{l}_0 \sim N_T\left(\mathbf{e}\eta - \boldsymbol{K}\boldsymbol{\beta}, \left(\boldsymbol{\Lambda}_\mathbf{y}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_\mathbf{y}\right)^{-1}\mathbf{I}_T\right)$ dan

$\boldsymbol{l}_i \sim N_T\left(\mathbf{e}\eta_i, \left(\boldsymbol{\Lambda}_\mathbf{y}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_\mathbf{y}\right)^{-1}\mathbf{I}_T\right)$

$$L(\omega) = \prod_{i=1}^{n} f\left(\boldsymbol{l}_{0_i}\right)$$

$$L(\omega) = \prod_{i=1}^{n}(2\pi)^{-\frac{T}{2}}\left|\left(\boldsymbol{\Lambda}_\mathbf{y}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_\mathbf{y}\right)^{-1}\mathbf{I}_T\right|^{-\frac{1}{2}} \times$$

$$\exp\left\{\frac{1}{2}\left(\boldsymbol{l}_{0_i} - (\mathbf{e}\eta + \boldsymbol{K}\boldsymbol{\beta})\right)^{'}\left(\boldsymbol{\Lambda}_\mathbf{y}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_\mathbf{y}\right)^{-1}\mathbf{I}_T\left(\boldsymbol{l}_{0_i} - (\mathbf{e}\eta + \boldsymbol{K}\boldsymbol{\beta})\right)\right\}$$

$$L(\omega)=(2\pi)^{-\frac{nT}{2}}\left|\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right|^{-\frac{n}{2}}\times$$

$$\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\left(\boldsymbol{l}_{0_i}-(\mathbf{e}\eta-\boldsymbol{K\beta})\right)^{'}\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)\mathbf{I}_T\left(\boldsymbol{l}_{0_i}-(\mathbf{e}\eta-\boldsymbol{K\beta})\right)\right\}.$$

c. Menentukan *odds ratio*

Di bawah populasi nilai $L(\hat{\Omega})$ adalah

$$L(\Omega)=(2\pi)^{-\frac{nT}{2}}\left|\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right|^{-\frac{n}{2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{l}_i-\mathbf{e}\eta)^{'}\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)\mathbf{I}_T(\boldsymbol{l}_i-\mathbf{e}\eta)\right\}$$

$$L(\hat{\Omega})=\frac{\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{l}_i-\mathbf{e}\eta)^{'}\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)\mathbf{I}_T(\boldsymbol{l}_i-\mathbf{e}\eta)\right\}}{(2\pi)^{\frac{nT}{2}}\left|\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right|^{\frac{n}{2}}}$$

$$L(\hat{\Omega})=\frac{\exp\left\{-\frac{1}{2}tr\left(\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)\mathbf{I}_T\sum_{i=1}^{n}(\boldsymbol{l}_i-\mathbf{e}\eta)^{'}(\boldsymbol{l}_i-\mathbf{e}\eta)^{'}\right)\right\}}{(2\pi)^{\frac{nT}{2}}\left|\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right|^{\frac{n}{2}}}$$

Karena $(\boldsymbol{l}_i-\mathbf{e}\eta)^{'}(\boldsymbol{l}_i-\mathbf{e}\eta)^{'}=\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T$ maka

$\sum_{i=1}^{n}(\boldsymbol{l}_i-\mathbf{e}\eta)^{'}(\boldsymbol{l}_i-\mathbf{e}\eta)^{'}=n\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T$ sehingga

$$L(\hat{\Omega})=\frac{\exp\left\{-\frac{1}{2}tr\left(\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)\mathbf{I}_T n\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right)\right\}}{(2\pi)^{\frac{nT}{2}}\left|n^{-1}\sum_{i=1}^{n}(\boldsymbol{l}_i-\mathbf{e}\eta)^{'}(\boldsymbol{l}_i-\mathbf{e}\eta)^{'}\right|^{\frac{n}{2}}}$$

$$L(\hat{\Omega})=\frac{\exp\left\{-\frac{1}{2}tr(n\mathbf{I}_T)\right\}}{(2\pi)^{\frac{nT}{2}}\left|n^{-1}\sum_{i=1}^{n}(\boldsymbol{l}_i-\mathbf{e}\eta)^{'}(\boldsymbol{l}_i-\mathbf{e}\eta)^{'}\right|^{\frac{n}{2}}}$$

$$L(\hat{\Omega})=\frac{\exp\left\{-\frac{1}{2}nT\right\}}{(2\pi)^{\frac{nT}{2}}\left|n^{-1}\sum_{i=1}^{n}(\boldsymbol{l}_i-\mathbf{e}\eta)^{'}(\boldsymbol{l}_i-\mathbf{e}\eta)^{'}\right|^{\frac{n}{2}}}$$

Di bawah H_0 nilai $L(\hat{\omega})$ adalah

$$L(\hat{\omega}) = \frac{\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)\mathbf{I}_T\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right\}}{(2\pi)^{\frac{nT}{2}}\left|\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right|^{\frac{n}{2}}}$$

$$L(\hat{\omega}) = \frac{\exp\left\{-\frac{1}{2}tr\left(\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)\mathbf{I}_T\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right)\right\}}{(2\pi)^{\frac{nT}{2}}\left|\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right|^{\frac{n}{2}}}$$

Karena nilai $\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right) = \left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T$

maka $\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right) = n\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T$

sehingga

$$L(\hat{\omega}) = \frac{\exp\left\{-\frac{1}{2}tr\left(\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)\mathbf{I}_T\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right)\right\}}{(2\pi)^{\frac{nT}{2}}\left|\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right|^{\frac{n}{2}}}$$

$$L(\hat{\omega}) = \frac{\exp\left\{-\frac{1}{2}tr\left(\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)\mathbf{I}_T n\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right)\right\}}{(2\pi)^{\frac{nT}{2}}\left|n^{-1}\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right|^{\frac{n}{2}}}$$

$$L(\hat{\omega}) = \frac{\exp\left\{-\frac{1}{2}tr\left(n\mathbf{I}_T\right)\right\}}{(2\pi)^{\frac{nT}{2}}\left|n^{-1}\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right|^{\frac{n}{2}}}$$

$$L(\hat{\omega}) = \frac{\exp\left\{-\frac{1}{2}nT\right\}}{(2\pi)^{\frac{nT}{2}}\left|n^{-1}\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right|^{\frac{n}{2}}}$$

d. Nilai *odds ratio* adalah

$$\Lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})}$$

$$\Lambda = \frac{\left| n^{-1}\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_{0_i} - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right|^{\frac{n}{2}}}{\left| n^{-1}\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \mathbf{e}\eta\right)'\left(\boldsymbol{l}_i - \mathbf{e}\eta\right)'\right|^{\frac{n}{2}}}$$

$$\Lambda^{-\frac{2}{n}} = \left| \frac{\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \mathbf{e}\eta\right)'\left(\boldsymbol{l}_i - \mathbf{e}\eta\right)'}{\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)} \right|$$

Jika terdapat matrik $\mathbf{C} = \left(\begin{array}{c|c} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \hline \mathbf{C}_{21} & \mathbf{C}_{22} \end{array}\right)$ maka

$$|\mathbf{C}| = |\mathbf{C}_{11}|\left|\mathbf{C}_{22} - \mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{C}_{12}\right| = |\mathbf{C}_{22}|\left|\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\right|$$

Misalkan terdapat matrik A dengan

$$|\mathbf{C}| = \left|\begin{array}{c|c} \sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right) & \sqrt{n}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)' \\ \hline \sqrt{n}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right) & -1 \end{array}\right|,$$ maka nilai

$|\mathbf{C}_{22}|\left|\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\right|$ adalah

$$= |-1|\left|\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right) - \sqrt{n}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'(-1)^{-1}\sqrt{n}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right|$$

$$= -\left|\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right) + n\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right|$$

Bentuk lain dari persamaan $\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \mathbf{e}\eta\right)'\left(\boldsymbol{l}_i - \mathbf{e}\eta\right)$ adalah sebagai berikut:

$$= \sum_{i=1}^{n}\left(\boldsymbol{l}_i - \mathbf{e}\eta\right)'\left(\boldsymbol{l}_i - \mathbf{e}\eta\right)$$

$$= \sum_{i=1}^{n}\left(\boldsymbol{l}_i - \mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} + \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)'\left(\boldsymbol{l}_i - \mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} + \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)$$

$$= \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) + \boldsymbol{K}\hat{\boldsymbol{\beta}} \right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) + \boldsymbol{K}\hat{\boldsymbol{\beta}} \right)$$

$$= \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right) - \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right)' \boldsymbol{K}\hat{\boldsymbol{\beta}} + \sum_{i=1}^{n} \left(\boldsymbol{K}\hat{\boldsymbol{\beta}} \right)' \left(\boldsymbol{K}\hat{\boldsymbol{\beta}} \right)$$

$$= \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right) + n \left(\boldsymbol{K}\hat{\boldsymbol{\beta}} \right)' \left(\boldsymbol{K}\hat{\boldsymbol{\beta}} \right)$$

$$\sum_{i=1}^{n} \left(\boldsymbol{l}_i - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right)' \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) = \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right)' \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right)$$

Karena $\sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right)' = 0$ maka

$$\sum_{i=1}^{n} \left(\boldsymbol{l}_i - \mathbf{e}\eta \right)' \left(\boldsymbol{l}_i - \mathbf{e}\eta \right) = \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right) + n \left(\boldsymbol{K}\hat{\boldsymbol{\beta}} \right)' \left(\boldsymbol{K}\hat{\boldsymbol{\beta}} \right)$$

Sehingga $\left| \mathbf{C}_{22} \right| \left| \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21} \right| = -\left| \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \mathbf{e}\eta \right)' \left(\boldsymbol{l}_i - \mathbf{e}\eta \right) \right|$.

Selanjutnya dicari nilai $\left| \mathbf{C}_{11} \right| \left| \mathbf{C}_{22} - \mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{C}_{12} \right|$, yaitu

$$= \left| \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right) \right| \times$$

$$\left| -1 - \sqrt{n} \left(\boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \left(\sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right) \right)^{-1} \sqrt{n} \left(\boldsymbol{K}\hat{\boldsymbol{\beta}} \right)' \right|$$

$$= -\left| \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right) \right| \times$$

$$\left| 1 + n \left(\boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \left(\sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right) \right)^{-1} \left(\boldsymbol{K}\hat{\boldsymbol{\beta}} \right)' \right|$$

dimana $\sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right) = n \left(\left(\boldsymbol{\Lambda}_{\mathrm{y}}' \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_{\mathrm{y}} \right)^{-1} \mathbf{I}_T \right)$,

sehingga

$$= -\left| \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right) \right| \left| 1 + n \left(\boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \left(n \left(\left(\boldsymbol{\Lambda}_{\mathrm{y}}' \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_{\mathrm{y}} \right)^{-1} \mathbf{I}_T \right) \right)^{-1} \left(\boldsymbol{K}\hat{\boldsymbol{\beta}} \right)' \right|$$

$$= -\left| \sum_{i=1}^{n} \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right)' \left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \right) \right| \left| 1 + \left(\boldsymbol{K}\hat{\boldsymbol{\beta}} \right) \left(\left(\boldsymbol{\Lambda}_{\mathrm{y}}' \boldsymbol{\Theta}_{\varepsilon*}^{-1} \boldsymbol{\Lambda}_{\mathrm{y}} \right)^{-1} \mathbf{I}_T \right)^{-1} \left(\boldsymbol{K}\hat{\boldsymbol{\beta}} \right)' \right|.$$

Oleh karena itu nilai dari $\left| \mathbf{C}_{11} \right| \left| \mathbf{C}_{22} - \mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{C}_{12} \right|$ adalah

$$\left|\mathbf{C}_{11}\right|\left|\mathbf{C}_{22}-\mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{C}_{12}\right| = -\left|\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right|\times$$

$$\left|1+\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\left(\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right)^{-1}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)^{'}\right|$$

Selanjutnya menyamakan

$$\left|\mathbf{C}_{11}\right|\left|\mathbf{C}_{22}-\mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{C}_{12}\right| = \left|\mathbf{C}_{22}\right|\left|\mathbf{C}_{11}-\mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\right|$$

$$-\left|\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\mathbf{e}\eta\right)^{'}\left(\boldsymbol{l}_i-\mathbf{e}\eta\right)\right| = -\left|\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)\right|\times$$

$$\left|1+\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\left(\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right)^{-1}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)^{'}\right|$$

$$\left|\frac{\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\mathbf{e}\eta\right)^{'}\left(\boldsymbol{l}_i-\mathbf{e}\eta\right)}{\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)}\right| = \left|1+\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\left(\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right)^{-1}\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)^{'}\right|. \quad (7.2)$$

Diketahui bahwa $\boldsymbol{l}_0 \sim N_T\left(\mathbf{e}\eta-\boldsymbol{K}\boldsymbol{\beta},\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right)$. Jika $\boldsymbol{l}^* = \boldsymbol{l}_0-\mathbf{e}\eta$

maka $\boldsymbol{E}\left(\boldsymbol{l}^*\right) = E\left(\boldsymbol{l}_0-\mathbf{e}\eta\right) = E\left(\boldsymbol{l}_0\right)-\mathbf{e}\eta = \mathbf{e}\eta-\boldsymbol{K}\boldsymbol{\beta}-\mathbf{e}\eta = -\boldsymbol{K}\boldsymbol{\beta}$

$\text{var}\left(\boldsymbol{l}^*\right) = \text{var}\left(\boldsymbol{l}_0-\mathbf{e}\eta\right) = \text{var}\left(\boldsymbol{l}_0\right) = \left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathrm{I}_T$. Oleh karena itu, didapatkan distribusi dari $\boldsymbol{l}^*$ yaitu $\boldsymbol{l}^* \sim N\left(-\boldsymbol{K}\boldsymbol{\beta},\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\right)$.

Di bawah H_0 : $\beta_1 = \beta_2 = \cdots = \beta_p = 0$, didapatkan $\left(-\boldsymbol{K}\hat{\boldsymbol{\beta}}-\mathbf{0}\right)^{'}\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\left(-\boldsymbol{K}\hat{\boldsymbol{\beta}}-\mathbf{0}\right) = T^2$ atau $\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)^{'}\left(\boldsymbol{\Lambda}_{\mathbf{y}}^{'}\boldsymbol{\Theta}_{\varepsilon*}^{-1}\boldsymbol{\Lambda}_{\mathbf{y}}\right)^{-1}\mathbf{I}_T\left(\boldsymbol{K}\hat{\boldsymbol{\beta}}\right) = T^2$ maka persamaan (7.2) menjadi

$$\left|\frac{\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\mathbf{e}\eta\right)^{'}\left(\boldsymbol{l}_i-\mathbf{e}\eta\right)}{\sum_{i=1}^{n}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)^{'}\left(\boldsymbol{l}_i-\left(\mathbf{e}\eta-\boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)}\right| = \left|1+T^2\right|$$

$$\Lambda^{-\frac{2}{n}} = 1+T^2$$

$$T^2 = \Lambda^{-\frac{2}{n}}-1$$

Sedangkan distribusi T^2 *Hotelling* adalah $T^2 \sim \frac{(n-1)p}{(n-p)}F_{p,n-p}$.

Daerah penolakan H_0 adalah

$$\alpha = P\left(T^2 > \frac{(n-1)p}{(n-p)} F_{(p,n-p(\alpha))}\right)$$

$$\alpha = P\left(\Lambda^{-\frac{2}{n}} - 1 > \frac{(n-1)p}{(n-p)} F_{(p,n-p(\alpha))}\right)$$

$$\alpha = P\left(\left(\Lambda^{-\frac{2}{n}} - 1\right)\frac{(n-p)}{(n-1)p} > F_{(p,n-p(\alpha))}\right)$$

Uji *maximum likelihood ratio* untuk model SERM-SEM berdasarkan uji F dengan statitik uji $\left(\Lambda^{-\frac{2}{n}} - 1\right)\frac{(n-p)}{(n-1)p}$ dimana

$$\Lambda^{-\frac{2}{n}} = \left|\frac{\sum_{i=1}^{n}(\boldsymbol{l}_i - \mathbf{e}\eta)'(\boldsymbol{l}_i - \mathbf{e}\eta)}{\sum_{i=1}^{n}\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)'\left(\boldsymbol{l}_i - \left(\mathbf{e}\eta - \boldsymbol{K}\hat{\boldsymbol{\beta}}\right)\right)}\right|$$

Pada tingkat signifikansi α hipotesis nol ditolak jika $\left(\Lambda^{-\frac{2}{n}} - 1\right)\frac{(n-p)}{(n-1)p} > F_{(p,n-p(\alpha))}$.

LAMPIRAN

Lampiran 1. Definisi, Teorema, Lemma, dan Corollary yang Digunakan untuk Pembuktian.

A. Nilai Ekspektasi dan Variansi dari Variabel Random

Definisi A.1 (Sahoo, 2013)

Diberikan X adalah variabel random kontinu dengan ruang hasil R_X dan pdf *f*(*x*), maka ekspektasi dari X didefiniskan dengan $E(X)=\int_{-\infty}^{\infty} xf(x)\,dx$

Teorema A.**1** (Sahoo, 2013)

Diberikan X adalah variabel random dengan pdf *f*(*x*). Jika *a* dan *b* adalah bilangan real maka didapatkan sifat-sifat:

a. $E(a)=a$

b. $E(bX)=bE(X)$

c. $E(a+bX)=a+bE(X)$

Definisi A.2 (Sahoo, 2013)

Diberikan X adalah variabel random dengan mean E(X) atau μ_X. Varians dari X dinotasikan dengan Var(X) atau σ_X^2 dan didefiniskan dengan $Var(X)=E\left(\left(X-E(X)\right)^2\right)$.

Teorema A.2 (Sahoo, 2013)

Jika X adalah variabel random dengan mean μ_X dan varians σ_X^2 maka $\sigma_X^2=E\left(X^2\right)-\left(\mu_X\right)^2$.

Teorema A.3 (Sahoo, 2013)

Jika X adalah variabel random dengan mean μ_X dan varians σ_X^2 atau Var(X) serta *a* dan *b* adalah bilangan real maka didapatkan sifat-sifat:

a. $Var(a)=0$;

b. $Var(bX)=b^2Var(X)$;

c. $Var(a+bX) = b^2 Var(X)$

B. Distribusi Matrik Normal Variat

Definisi B.1 (Gupta dan Nagar, 2000)

Distribusi normal variat matrik muncul ketika pengambilan sampel populasi normal multivariat. Diberikan $x_1,...,x_N$ sampel random ukuran N dari $N_P(\mu,\Sigma)$. Mendefinisikan matrik random pengamatan sebagai: $X = \begin{bmatrix} x_{11} & \cdots & x_{1N} \\ x_{21} & \cdots & x_{2N} \\ \vdots & \ddots & \vdots \\ x_{p1} & \cdots & x_{pN} \end{bmatrix} = (x_1,...,x_N) = \begin{bmatrix} x_1^{*'} \\ \vdots \\ x_p^{*'} \end{bmatrix}$ maka $X' \sim N_{N,p}(e\mu, \mathrm{I}_N \otimes \Sigma)$ dimana $e_{N\times 1} = (1,\ldots,1)'$.

Teorema B.1 (Gupta dan Nagar, 2000)

Jika $X \sim N_{p,n}(M,\Sigma\otimes\Psi)$ maka $X' \sim N_{p,n}(M,\Psi\otimes\Sigma)$

Teorema **B.2** (Gupta dan Nagar, 2000)

Jika $X \sim N_{p,n}(M,\Sigma\otimes\Psi)$ maka fungsi karakteristik dari **X** adalah $_{\phi X}(\mathbf{Z}) = etr\left(\iota\mathbf{Z}'\mathbf{M} - 1/2\mathbf{Z}'\Sigma\mathbf{Z}\Psi\right)$ dengan $\iota = \sqrt{-1}$

C. Sifat-sifat Matrik serta Turunan Matrik dan Vektor

Teorema C.1 (Gupta dan Nagar, 2000)

$tr(\mathbf{AB}) = tr(\mathbf{BA})$ dengan $\mathbf{A}_{p\times q}$ dan $\mathbf{B}_{q\times p}$

Teorema turunan matrik dan vektor:

Sifat turunan matrik C.2. $\dfrac{\partial \ln|\mathbf{X}|}{\partial x} = Tr\left(\mathbf{X}^{-1}\dfrac{\partial \mathbf{X}}{\partial x}\right)$

Sifat turunan matrik C.3. $\dfrac{\partial\left(\mathbf{X}^{-1}\right)}{\partial x} = -\mathbf{X}^{-1}\left(\dfrac{\partial \mathbf{X}}{\partial x}\right)\mathbf{X}^{-1}$

Sifat turunan matrik C.4. $\dfrac{\partial \mathbf{A}_{(x)}^T\mathbf{A}_{(x)}}{\partial x} = 2\left(\dfrac{\partial \mathbf{A}_{(x)}}{\partial x}\right)^T \mathbf{A}_{(x)}$

Sifat turunan matrik C.5. $\frac{\partial}{\partial \mathbf{s}}(\mathbf{x} - \mathbf{A}\mathbf{s})^T (\mathbf{x} - \mathbf{A}\mathbf{s}) = -2\mathbf{A}^T (\mathbf{x} - \mathbf{A}\mathbf{s})$

Sifat turunan matrik C.6. $\frac{\partial}{\partial \mathbf{x}} \mathbf{s}' \mathbf{A}\mathbf{r} = \left[\frac{\partial \mathbf{s}}{\partial \mathbf{x}}\right]' \mathbf{A}\mathbf{r} + \left[\frac{\partial \mathbf{r}}{\partial \mathbf{x}}\right] \mathbf{A}' \mathbf{s}$

Sifat turunan matrik C.7. $\frac{\partial \mathbf{A}_{(x)}^T \mathbf{B}_{(x)}}{\partial x} = \left[\frac{\partial \mathbf{A}_{(x)}}{\partial x}\right]^T \mathbf{B}_{(x)} + \left[\frac{\partial \mathbf{B}_{(x)}}{\partial x}\right]^T \mathbf{A}_{(x)}$

Sifat turunan matrik C.8.

$$\frac{\partial \mathbf{A}_{(x)}^T \mathbf{B}_{(x)}}{\partial x^T} = \left(\frac{\partial \mathbf{B}_{(x)}^T \mathbf{A}_{(x)}}{\partial x}\right)^T = \left[\left(\frac{\partial \mathbf{B}_{(x)}}{\partial x}\right)^T \mathbf{A}_{(x)} + \left(\frac{\partial \mathbf{A}_{(x)}}{\partial x}\right)^T \mathbf{B}_{(x)}\right]^T$$

$$= \mathbf{A}_{(x)}^T \frac{\partial \mathbf{B}_{(x)}}{\partial x^T} + \mathbf{B}_{(x)}^T \frac{\partial \mathbf{A}_{(x)}}{\partial x}$$

Lampiran 2. Pembuktian Matrik $\left(\underset{p\times A}{\mathbf{\Lambda}_{\mathbf{x}}'}\underset{A\times A}{\mathbf{\Theta}_{\boldsymbol{\delta}}^{-1}}\underset{A\times p}{\mathbf{\Lambda}_{\mathbf{x}}}\right)$ Adalah Matrik Diagonal dengan Elemen Selain Diagonal Utama Adalah Nol.

Masing-masing matrik $\left(\underset{p\times A}{\mathbf{\Lambda}_{\mathbf{x}}'}\underset{A\times A}{\mathbf{\Theta}_{\boldsymbol{\delta}}^{-1}}\underset{A\times p}{\mathbf{\Lambda}_{\mathbf{x}}}\right)$ dibuat matrik partisi untuk mempermudah pembuktian. Matrik partisi untuk $\underset{A\times p}{\mathbf{\Lambda}_{\mathbf{x}}}$ adalah:

$$\underset{A\times p}{\mathbf{\Lambda}_{\mathbf{x}}}=\begin{bmatrix} \lambda_{(1)1} & 0 & \cdots & 0 \\ \lambda_{(2)1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{(a_1)1} & 0 & \cdots & 0 \\ 0 & \lambda_{(1)2} & \cdots & 0 \\ 0 & \lambda_{(2)2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ & \lambda_{(a_2)2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{(1)p} \\ 0 & 0 & \cdots & \lambda_{(2)p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{(a_p)p} \end{bmatrix}=\underset{p\times 1}{\begin{bmatrix} \underset{a_1\times p}{\mathbf{A}_1} \\ \hline \underset{a_2\times p}{\mathbf{A}_2} \\ \hline \vdots \\ \hline \underset{a_p\times p}{\mathbf{A}_p} \end{bmatrix}}$$

Matrik partisi untuk $\underset{A\times A}{\mathbf{\Theta}_{\boldsymbol{\delta}}}$ adalah:

$$\underset{A\times A}{\mathbf{\Theta}_{\boldsymbol{\delta}}}=\left(\sigma^2_{\delta(1)1},\sigma^2_{\delta(2)1},\cdots,\sigma^2_{\delta(a1)1},\sigma^2_{\delta(1)2},\sigma^2_{\delta(2)2},\cdots,\sigma^2_{\delta(a2)2},\cdots\sigma^2_{\delta(1)p},\sigma^2_{\delta(2)p},\cdots,\sigma^2_{\delta(ap)p}\right)$$

$$=\underset{p\times p}{\left[\begin{array}{c|c|c|c} \underset{a_1\times a_1}{\mathbf{B}_1} & \underset{a_1\times a_2}{\mathbf{0}} & \cdots & \underset{a_1\times a_p}{\mathbf{0}} \\ \hline \underset{a_2\times a_1}{\mathbf{0}} & \underset{a_2\times a_2}{\mathbf{B}_2} & \cdots & \underset{a_2\times a_p}{\mathbf{0}} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \underset{a_p\times a_1}{\mathbf{0}} & \underset{a_p\times a_2}{\mathbf{0}} & \cdots & \underset{a_p\times a_p}{\mathbf{B}_p} \end{array}\right]}$$

$$\underset{A\times A}{\boldsymbol{\Theta}_{\boldsymbol{\delta}}} = \left[\begin{array}{c|c|c|c} \underset{a_1\times a_1}{\mathbf{B}_1} & \underset{a_1\times a_2}{\mathbf{0}} & \cdots & \underset{a_1\times a_p}{\mathbf{0}} \\ \hline \underset{a_2\times a_1}{\mathbf{0}} & \underset{a_2\times a_2}{\mathbf{B}_2} & \cdots & \underset{a_2\times a_p}{\mathbf{0}} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \underset{a_p\times a_1}{\mathbf{0}} & \underset{a_p\times a_2}{\mathbf{0}} & \cdots & \underset{a_p\times a_p}{\mathbf{B}_p} \end{array}\right]_{p\times p} = \left[\begin{array}{c|c} \underset{a\times a}{\mathbf{C}} & \underset{a\times(p-a)}{\mathbf{0}} \\ \hline \underset{(p-a)\times a}{\mathbf{0}} & \underset{(p-a)\times(p-a)}{\mathbf{D}} \end{array}\right]_{p\times p}$$

Jika ada matrik partisi $\mathbf{C} = \left(\begin{array}{c|c} \mathbf{C}_1 & \mathbf{C}_2 \\ \hline \mathbf{C}_3 & \mathbf{C}_4 \end{array}\right)$ maka matrik inversnya adalah

$\mathbf{C}^{-1} = \left(\begin{array}{c|c} \mathbf{C}_{E1} & \mathbf{C}_{E2} \\ \hline \mathbf{C}_{E3} & \mathbf{C}_{E4} \end{array}\right)$ dimana $\mathbf{C}_{E1} = \left(\mathbf{C}_1 - \mathbf{C}_2\mathbf{C}_4^{-1}\mathbf{C}_3\right)^{-1}$, $\mathbf{C}_{E2} = \left(-\mathbf{C}_{E1}\mathbf{C}_2\mathbf{C}_4^{-1}\right)$, $\mathbf{C}_{E3} = \left(-\mathbf{C}_4^{-1}\mathbf{C}_3\mathbf{C}_{E1}\right)$, dan $\mathbf{C}_{E4} = \left(\mathbf{C}_4^{-1} - \mathbf{C}_4^{-1}\mathbf{C}_3\mathbf{C}_{E2}\right)$.

Maka $\underset{A\times A}{\boldsymbol{\Theta}_{\boldsymbol{\delta}}^{-1}} = \left[\begin{array}{c|c} \underset{a\times a}{\mathbf{C}} & \underset{a\times(p-a)}{\mathbf{0}} \\ \hline \underset{(p-a)\times a}{\mathbf{0}} & \underset{(p-a)\times(p-a)}{\mathbf{D}} \end{array}\right]_{p\times p}^{-1} = \left[\begin{array}{c|c} \underset{a\times a}{\mathbf{C}^{-1}} & \underset{a\times(p-a)}{\mathbf{0}} \\ \hline \underset{(p-a)\times a}{\mathbf{0}} & \underset{(p-a)\times(p-a)}{\mathbf{D}^{-1}} \end{array}\right]_{p\times p}$

$$\underset{A\times A}{\boldsymbol{\Theta}_{\boldsymbol{\delta}}^{-1}} = \left[\begin{array}{c|c|c|c} \underset{a_1\times a_1}{\mathbf{B}_1^{-1}} & \underset{a_1\times a_2}{\mathbf{0}} & \cdots & \underset{a_1\times a_p}{\mathbf{0}} \\ \hline \underset{a_2\times a_1}{\mathbf{0}} & \underset{a_2\times a_2}{\mathbf{B}_2^{-1}} & \cdots & \underset{a_2\times a_p}{\mathbf{0}} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \underset{a_p\times a_1}{\mathbf{0}} & \underset{a_p\times a_2}{\mathbf{0}} & \cdots & \underset{a_p\times a_p}{\mathbf{B}_P^{-1}} \end{array}\right]_{p\times p}$$

Dimana $\underset{a_1\times a_1}{\mathbf{B}_1}$ adalah matrik diagonal dengan elemen matrik selain elemen diagonal utama adalah nol, sehingga $\underset{a_1\times a_1}{\mathbf{B}_1^{-1}}$ adalah matrik diagonal diagonal dengan elemen matrik selain elemen diagonal utama adalah nol.

$$\underset{p\times A}{\boldsymbol{\Lambda}_{\mathbf{x}}'}\underset{A\times A}{\boldsymbol{\Theta}_{\boldsymbol{\delta}}^{-1}}\underset{A\times p}{\boldsymbol{\Lambda}_{\mathbf{x}}}=\underset{1\times p}{\left[\begin{array}{c|c|c|c}\underset{p\times a_1}{\mathbf{A}_1'} & \underset{p\times a_2}{\mathbf{A}_2'} & \cdots & \underset{p\times a_p}{\mathbf{A}_p'}\end{array}\right]}\underset{p\times p}{\left[\begin{array}{c|c|c|c}\underset{a_1\times a_1}{\mathbf{B}_1^{-1}} & \underset{a_1\times a_2}{\mathbf{0}} & \cdots & \underset{a_1\times a_p}{\mathbf{0}}\\ \hline \underset{a_2\times a_1}{\mathbf{0}} & \underset{a_2\times a_2}{\mathbf{B}_2^{-1}} & \cdots & \underset{a_2\times a_p}{\mathbf{0}}\\ \hline \vdots & \vdots & \ddots & \vdots\\ \hline \underset{a_p\times a_1}{\mathbf{0}} & \underset{a_p\times a_2}{\mathbf{0}} & \cdots & \underset{a_p\times a_p}{\mathbf{B}_P^{-1}}\end{array}\right]}\underset{p\times 1}{\left[\begin{array}{c}\underset{a_1\times p}{\mathbf{A}_1}\\ \hline \underset{a_2\times p}{\mathbf{A}_2}\\ \hline \vdots\\ \hline \underset{a_p\times p}{\mathbf{A}_p}\end{array}\right]}$$

$$\underset{p\times A}{\boldsymbol{\Lambda}_{\mathbf{x}}'}\underset{A\times A}{\boldsymbol{\Theta}_{\boldsymbol{\delta}}^{-1}}\underset{A\times p}{\boldsymbol{\Lambda}_{\mathbf{x}}}=\left[\underset{p\times a_1}{\mathbf{A}_1'}\underset{a_1\times a_1}{\mathbf{B}_1^{-1}}\underset{a_1\times p}{\mathbf{A}_1}+\underset{p\times a_2}{\mathbf{A}_2'}\underset{a_2\times a_2}{\mathbf{B}_2^{-1}}\underset{a_2\times p}{\mathbf{A}_2}+\ldots+\underset{p\times a_p}{\mathbf{A}_p'}\underset{a_p\times a_p}{\mathbf{B}_p^{-1}}\underset{a_p\times p}{\mathbf{A}_p}\right]$$

$$\underset{p\times a_1}{\mathbf{A}_1'}\underset{a_1\times a_1}{\mathbf{B}_1^{-1}}\underset{a_1\times p}{\mathbf{A}_1}=\underset{p\times a_1}{\begin{bmatrix}\lambda_{(1)1} & \lambda_{(2)1} & \cdots & \lambda_{(a_1)1}\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}}\underset{a_1\times a_1}{\begin{bmatrix}b_1 & 0 & \cdots & 0\\ 0 & b_1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & b_{a_1}\end{bmatrix}}\underset{a_1\times p}{\begin{bmatrix}\lambda_{(1)1} & 0 & \cdots & 0\\ \lambda_{(2)1} & b_1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_{(a_1)1} & 0 & \cdots & 0\end{bmatrix}}$$

$$\underset{p\times a_1}{\mathbf{A}_1'}\underset{a_1\times a_1}{\mathbf{B}_1^{-1}}\underset{a_1\times p}{\mathbf{A}_1}=\underset{p\times a_1}{\begin{bmatrix}\lambda_{(1)1}b_1 & 0 & \cdots & 0\\ 0 & \lambda_{(2)1}b_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_{(a_1)1}b_{a_1}\end{bmatrix}}\underset{a_1\times p}{\begin{bmatrix}\lambda_{(1)1} & 0 & \cdots & 0\\ \lambda_{(2)1} & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ \lambda_{(a_1)1} & 0 & \cdots & 0\end{bmatrix}}$$

$$\underset{p\times a_1}{\mathbf{A}_1'}\underset{a_1\times a_1}{\mathbf{B}_1^{-1}}\underset{a_1\times p}{\mathbf{A}_1}=\underset{p\times p}{\begin{bmatrix}\lambda_{(1)1}b_1\lambda_{(1)1} & 0 & \cdots & 0\\ 0 & \lambda_{(2)1}b_2\lambda_{(2)1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_{(a_1)1}b_{a_1}\lambda_{(a_1)1}\end{bmatrix}}$$

sehingga didapatkan matrik $\underset{p\times a_1}{\mathbf{A}_1'}\underset{a_1\times a_1}{\mathbf{B}_1^{-1}}\underset{a_1\times p}{\mathbf{A}_1}$ adalah matrik diagonal dengan elemen matrik selain elemen diagonal utama adalah nol, demikian juga dengan matrik $\underset{p\times a_2}{\mathbf{A}_1'}\underset{a_2\times a_2}{\mathbf{B}_1^{-1}}\underset{a_2\times p}{\mathbf{A}_1}$ serta matrik . $\underset{p\times a_p}{\mathbf{A}_1'}\underset{a_p\times a_p}{\mathbf{B}_1^{-1}}\underset{a_p\times p}{\mathbf{A}_1}$

Oleh karena itu dapat disimpulkan bahwa matrik $\left(\underset{p\times A}{\boldsymbol{\Lambda}_{\mathbf{x}}'}\underset{A\times A}{\boldsymbol{\Theta}_{\boldsymbol{\delta}}^{-1}}\underset{A\times p}{\boldsymbol{\Lambda}_{\mathbf{x}}}\right)$ adalah matrik diagonal dengan elemen selain diagonal utama adalah nol.

Lampiran 3. Pembuktian Variabel Regressor *Wl* Berkorelasi dengan *Error* (*u*) Atau $\operatorname{cov}(\mathbf{W}l,\mathbf{u})\neq 0$

Parameter pada model SEM spasial di atas tidak dapat diduga secara konsisten dengan menggunakan metode OLS karena terdapat kasus bahwa variabel regressor **W*l*** berkorelasi dengan *error* u, sehingga akan dibuktikan bahwa $E\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times 1}{l}\right)\underset{1\times T}{u'}\right]\neq 0$ atau elemen spasial *autoregressive* (vektor ***Wl***) saling berhubungan dengan *error* (**u**).

Pada persamaan (2.35) dan berdasar asumsi 5 bahwa ε_i berdistribusi identik (bersifat independen) sehingga $E(\varepsilon_i)=0$, $E\left(\varepsilon_i\varepsilon_j\right)=E(\varepsilon_i)E\left(\varepsilon_j\right)=0$. Berikut ini akan dibuktikan terlebih dahulu bahwa $E\left(\varepsilon_i^2\right)=\sigma_\varepsilon^2$

$$E\left(\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}\right)=E\begin{pmatrix}\varepsilon_1^2 & \varepsilon_1\varepsilon_2 & \cdots & \varepsilon_1\varepsilon_n\\ \varepsilon_2\varepsilon_1 & \varepsilon_2^2 & \cdots & \varepsilon_2\varepsilon_n\\ \vdots & \vdots & \ddots & \vdots\\ \varepsilon_n\varepsilon_1 & \varepsilon_n\varepsilon_2 & \cdots & \varepsilon_n^2\end{pmatrix}$$

$$E\left(\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}\right)=\begin{pmatrix}E\left(\varepsilon_1^2\right) & E(\varepsilon_1\varepsilon_2) & \cdots & E(\varepsilon_1\varepsilon_n)\\ E(\varepsilon_2\varepsilon_1) & E\left(\varepsilon_2^2\right) & \cdots & E(\varepsilon_2\varepsilon_n)\\ \vdots & \vdots & \ddots & \vdots\\ E(\varepsilon_n\varepsilon_1) & E(\varepsilon_n\varepsilon_2) & \cdots & E\left(\varepsilon_n^2\right)\end{pmatrix}$$

$$E\left(\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}\right)=\begin{pmatrix}\sigma_\varepsilon^2 & 0 & \cdots & 0\\ 0 & \sigma_\varepsilon^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \sigma_\varepsilon^2\end{pmatrix}$$

$$E\left(\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}\right)=\sigma_\varepsilon^2\begin{pmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}=\sigma_\varepsilon^2\mathbf{I}$$

Nilai $\underset{T\times 1}{\mathbf{u}}=\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{M}}\right)^{-1}\underset{T\times 1}{\boldsymbol{\varepsilon}}$ maka

$$E\left(\underset{T\times 1}{\mathbf{u}}\underset{1\times T}{\mathbf{u}'}\right)=E\left[\left\{\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{M}}\right)^{-1}\underset{T\times 1}{\boldsymbol{\varepsilon}}\right\}\left\{\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{M}}\right)^{-1}\underset{T\times 1}{\boldsymbol{\varepsilon}}\right\}'\right]$$

$$E\left(\underset{T\times 1}{\mathbf{u}}\underset{1\times T}{\mathbf{u}'}\right)=E\left[\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{M}}\right)^{-1}\underset{T\times 1}{\boldsymbol{\varepsilon}}\underset{T\times 1}{\boldsymbol{\varepsilon}'}\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{M}'}\right)^{-1}\right]$$

$$E\left(\underset{T\times 1}{\mathbf{u}}\underset{1\times T}{\mathbf{u}'}\right)=E\left(\underset{T\times 1}{\boldsymbol{\varepsilon}}\underset{1\times T}{\boldsymbol{\varepsilon}'}\right)\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{M}}\right)^{-1}\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{M}'}\right)^{-1}$$

$$E\left(\underset{T\times 1}{\mathbf{u}}\underset{1\times T}{\mathbf{u}'}\right)=\sigma^2\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{M}}\right)^{-1}\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{M}'}\right)^{-1}$$

$$\operatorname{cov}\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times 1}{l}\right),\underset{T\times 1}{\mathbf{u}}\right]=E\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times 1}{l}\right),\underset{1\times T}{\mathbf{u}'}\right]-E\left(\underset{T\times T}{\mathbf{W}}\underset{T\times 1}{l}\right)E\left(\underset{1\times T}{\mathbf{u}'}\right)$$

Karena $E(\boldsymbol{\varepsilon})=0, E(\mathbf{u})=0, E\left(\boldsymbol{\varepsilon}_i^2\right)=\sigma_\varepsilon^2$ maka

$$\operatorname{cov}\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times 1}{l}\right),\underset{T\times 1}{\mathbf{u}}\right]=E\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times 1}{l}\right),\underset{1\times T}{\mathbf{u}'}\right]$$

$$\operatorname{cov}\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times 1}{l}\right),\underset{T\times 1}{\mathbf{u}}\right]=\underset{T\times T}{\mathbf{W}}E\left\{\left[\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)^{-1}\underset{T\times p}{\mathbf{K}}\underset{p\times 1}{\boldsymbol{\beta}}+\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)^{-1}\underset{T\times 1}{\mathbf{u}}\right]\underset{1\times T}{\mathbf{u}'}\right\}$$

$$\operatorname{cov}\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times 1}{l}\right),\underset{T\times 1}{\mathbf{u}}\right]=\underset{T\times T}{\mathbf{W}}E\left[\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)^{-1}\underset{T\times p}{\mathbf{K}}\underset{p\times 1}{\boldsymbol{\beta}}\underset{1\times T}{\mathbf{u}'}+\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)^{-1}\underset{T\times 1}{\mathbf{u}}\underset{1\times T}{\mathbf{u}'}\right]$$

$$\operatorname{cov}\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times 1}{l}\right),\underset{T\times 1}{\mathbf{u}}\right]=\underset{T\times T}{\mathbf{W}}\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)^{-1}\underset{T\times p}{\mathbf{K}}\underset{p\times 1}{\boldsymbol{\beta}}E\left(\underset{1\times T}{\mathbf{u}'}\right)+\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)^{-1}E\left(\underset{T\times 1}{\mathbf{u}}\underset{1\times T}{\mathbf{u}'}\right)$$

$$\operatorname{cov}\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times 1}{l}\right),\underset{T\times 1}{\mathbf{u}}\right]=\sigma^2\underset{T\times T}{\mathbf{W}}\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)^{-1}\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{M}}\right)^{-1}\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{M}'}\right)^{-1}$$

Sehingga $\operatorname{cov}\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times 1}{l}\right),\underset{T\times 1}{\mathbf{u}}\right]=E\left[\underset{T\times T}{\mathbf{W}}\underset{T\times 1}{l},\underset{1\times T}{\mathbf{u}'}\right]\neq 0$

Karena didapatkan pembuktian bahwa $E\left[\underset{T\times T}{\mathbf{W}}\underset{T\times 1}{l},\underset{1\times T}{\mathbf{u}'}\right]\neq 0$ maka dapat disimpulkan bahwa elemen spasial *autoregressive* (vektor *Wl*) berkorelasi dengan *error* (u).

Lampiran 4. Pembuktian Matrik $\mathbf{H} = (\mathbf{K}|\mathbf{WK})$ Merupakan Variabel Instrumen yang Valid

Agar mendapatkan estimator yang konsisten, maka dibutuhkan metode GMM, dimana langkah awalnya membutuhkan variabel instrumen. Oleh karena itu dibutuhkan variabel-variabel instrumen yang memenuhi kriteria sebagai berikut: Misalkan $\mathbf{H}_{nxp}$ merupakan matrik variabel instrumen yang berisi p variabel-variabel instrumen. Maka **H** harus mempunyai sifat:

1. Tidak berkorelasi dengan ***u*** (*error*).
2. Hanya mengandung variabel minimal sebanyak k atau $p \geq k$ sedemikian sehingga **H** berkorelasi dengan regresor ***Wl***

Model SEM spasial pada persamaan (2.35) akan menggunakan variabel instrumen yang merupakan penggabungan matrik ***K*** dan ***WK***, yaitu $\mathbf{H} = (\mathbf{K}|\mathbf{WK})$. Berikut adalah pembuktian variabel-variabel atau kolom-kolom pada matrik **H** tidak berkorelasi dengan **u** (*error*).

Misalkan $\boldsymbol{k}_t$ adalah vektor kolom dari matrik ***K*** yang mewakili nilai skor faktor ke-*t* dari ***K***. Pada model SEM spasial pada persamaan di atas diasumsikan bahwa ***K*** tidak berkorelasi dengan **u**, akibatnya setiap $\boldsymbol{k}_t$ tidak berkorelasi dengan **u**, sehingga setiap t berlaku $\operatorname{cov}(\boldsymbol{k}_t, \mathbf{u}) = 0$

Misalkan $(\mathbf{WK})$ adalah variabel ke-t dari ***WK***, maka berikut ini adalah pembuktian bahwa $(\mathbf{WK})$ tidak berkorelasi dengan **u** untuk sembarang *t*.

$$\operatorname{cov}\left[(\mathbf{W}\boldsymbol{K})_t, \mathbf{u}\right] = E\left[(\mathbf{W}\boldsymbol{K})_t \mathbf{u}'\right] - E\left[(\mathbf{W}\boldsymbol{K})_t\right] E\left[\mathbf{u}'\right]$$
$$= \mathbf{W}E\left[\boldsymbol{k}_t \mathbf{u}'\right] - \mathbf{W}E\left[\boldsymbol{k}_t\right] E\left[\mathbf{u}'\right]$$

$$\operatorname{cov}\left[(\mathbf{WK})_t, \mathbf{u}\right] = E\left[(\mathbf{WK})_t, \mathbf{u}'\right] - E\left[(\mathbf{WK})_t\right] E\left[\mathbf{u}'\right]$$
$$= \mathbf{W}E\left[\mathbf{k}_t, \mathbf{u}'\right] - \mathbf{W}E\left[\mathbf{k}_t\right] \mathbf{W}E\left[\mathbf{u}'\right]$$

$= \mathbf{W}\operatorname{cov}\left[\mathbf{k}_t, \mathbf{u}'\right]$ dengan nilai $\operatorname{cov}(\mathbf{k}_t, \mathbf{u}) = 0$

maka $\operatorname{cov}\left[(\mathbf{WK})_t, \mathbf{u}\right] = 0$ (a)

Hal ini merepresentasikan bahwa masing-masing skor faktor dari variabel laten eksogen *K* dan ***WK*** tidak berkorelasi dengan **u** atau variabel-variabel dalam matrik $\mathbf{H}=(\mathbf{K}|\mathbf{WK})$ tidak berkorelasi dengan **u**.

Berikut ini akan dibuktikan bahwa masing-masing variabel pada matrik variabel instrumen $\mathbf{H}=(\mathbf{K}|\mathbf{WK})$ berkorelasi dengan $\mathbf{W}l$

$$\begin{aligned}\text{cov}[k_t,\mathbf{W}l] &= E\left[k_t,(\mathbf{W}l)'\right]-E[k_t]E\left[(\mathbf{W}l)'\right]\\ &= E\left[k_t l'\mathbf{W}'\right]-E[k_t]E\left[l'\right]\mathbf{W}'\\ &= E\left[k_t l'\right]\mathbf{W}'-E[k_t]E\left[l'\right]\mathbf{W}'\\ &= \left\{E\left[k_t l'\right]-E[k_t]E\left[l'\right]\right\}\mathbf{W}'\end{aligned}$$

$=\text{cov}\left(k_t,l'\right)\mathbf{W}'$ karena $\text{cov}\left(k_t,l'\right)\neq 0$ dan $\mathbf{W}\neq 0$ maka

$$\text{cov}[k_t,\mathbf{W}l]\neq 0 \qquad \text{(b)}$$

Persamaan (b) membuktikan bahwa variabel-variabel pada *K* berkorelasi dengan $\mathbf{W}l$.

$$\begin{aligned}\text{cov}\left[(\mathbf{W}k)_t,\mathbf{W}l\right] &= E\left[(\mathbf{W}k)_t(\mathbf{W}l)'\right]-E\left[(\mathbf{W}k)_t\right]E\left[(\mathbf{W}l)'\right]\\ &= E\left[\mathbf{W}k_t l'\mathbf{W}'\right]-\mathbf{W}E[k_t]E\left[l'\right]\mathbf{W}'\\ &= \mathbf{W}E\left[k_t l'\right]\mathbf{W}'-\mathbf{W}E[k_t]E\left[l'\right]\mathbf{W}'\\ &= \mathbf{W}\left\{E\left[k_t l'\right]-E[k_t]E\left[l'\right]\right\}\mathbf{W}'\end{aligned}$$

$=\mathbf{W}\text{cov}\left(k_t,l'\right)\mathbf{W}'$ karena $\text{cov}\left(k_t,l'\right)\neq 0$ dan $\mathbf{W}\neq 0$ maka

$$\text{cov}\left[(\mathbf{W}k)_t,\mathbf{W}l\right]\neq 0 \qquad \text{(c)}$$

Persamaan (c) membuktikan bahwa variabel-variabel pada ***WK*** berkorelasi dengan $\mathbf{W}l$.

Karena variabel-variabel dalam *K* berkorelasi dengan $\mathbf{W}l$ sebagaimana hasil pada (b) dan ***WK*** berkorelasi dengan $\mathbf{W}l$ sebagaimana hasil pada (c) maka terbukti bahwa variabel-variabel dalam matrik $\mathbf{H}=(\mathbf{K}|\mathbf{WK})$ berkorelasi dengan $\mathbf{W}l$.

Dari uraian di atas telah dibuktikan bahwa matrik $\mathbf{H}=(\mathbf{K}|\mathbf{WK})$ tidak berkorelasi dengan **u** sebagaimana hasil pada persamaan (a) dan $\mathbf{H} = \left(\mathbf{K} \middle| \mathbf{WK}\right)$ berkorelasi dengan $\mathbf{W}\boldsymbol{l}$ sebagaimana hasil pada persamaan (b) dan (c) maka matrik $\mathbf{H} = \left(\mathbf{K} \middle| \mathbf{WK}\right)$ merupakan matrik variabel instrumen yang valid.

Lampiran 5. Turunan Pertama dan Kedua Fungsi *ln likelihood* $L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)$ pada Model SAR-SEM

1. Turunan pertama fungsi *ln likelihood* untuk $L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})$ terhadap λ

$$L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)=-\frac{T}{2}\ln|\pi|-\frac{1}{2}\ln|\boldsymbol{\Theta}|+\ln|\mathbf{A}|-\frac{1}{2}\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)$$

dengan $\underset{T\times T}{\boldsymbol{\Theta}}=\underset{T\times T}{\mathbf{A}}\underset{1\times 1}{\rho^{-1}}\underset{T\times T}{\mathbf{A}'}=\underset{1\times 1}{\rho^{-1}}\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{A}'}$; $\underset{1\times 1}{\rho}=\left(\underset{1\times B}{\boldsymbol{\Lambda}_y'}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon^*}^{-1}}\underset{B\times 1}{\boldsymbol{\Lambda}_y}\right)$;

$\underset{T\times T}{\mathbf{A}}=\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)$ dan $\underset{T\times 1}{\boldsymbol{\varepsilon}}=\underset{T\times T}{\mathbf{A}}\underset{T\times 1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}$

   a. Turunan pertama $\boldsymbol{\Theta}$ terhadap λ

   karena hasil perkalian $\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)'$ adalah matriks simetris

$$\underset{T\times T}{\boldsymbol{\Theta}}=\underset{T\times T}{\mathbf{A}}\underset{1\times 1}{p^{-1}}\underset{T\times T}{\mathbf{A}'}=\underset{1\times 1}{p^{-1}}\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{A}'}$$

$$\underset{T\times T}{\boldsymbol{\Theta}}=\underset{1\times 1}{p^{-1}}\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)'$$

$$\frac{\partial\boldsymbol{\Theta}}{\partial\lambda}=\frac{\partial\left(\underset{1\times 1}{p^{-1}}\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)'\right)}{\partial\lambda}=\underset{1\times 1}{p^{-1}}\frac{\partial\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)'}{\partial\lambda}$$

$$\frac{\partial\boldsymbol{\Theta}}{\partial\lambda}=\underset{1\times 1}{p^{-1}}\left(\frac{\partial\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)}{\partial\lambda}\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)'+\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)\frac{\partial\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)'}{\partial\lambda}\right)$$

$$\frac{\partial\boldsymbol{\Theta}}{\partial\lambda}=\underset{1\times 1}{p^{-1}}\left(\frac{-\partial\left(\lambda\underset{T\times T}{\mathbf{W}}\right)}{\partial\lambda}\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)'+\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)\frac{-\left(\partial\left(\lambda\underset{T\times T}{\mathbf{W}}\right)\right)'}{\partial\lambda}\right)$$

$$\frac{\partial\boldsymbol{\Theta}}{\partial\lambda}=\underset{1\times 1}{p^{-1}}\left(-\underset{T\times T}{\mathbf{W}}\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)'+\left(\underset{T\times T}{\mathbf{I}}-\lambda\underset{T\times T}{\mathbf{W}}\right)\left(-\underset{T\times T}{\mathbf{W}}\right)'\right)$$

$$\frac{\partial \mathbf{\Theta}}{\partial \lambda_{T\times T}} = -\underset{1\times 1}{p^{-1}}\left(\underset{T\times T}{\mathbf{W}}\left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)^{'} + \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)\underset{T\times T}{\mathbf{W}^{'}}\right)$$

$$\frac{\partial \mathbf{\Theta}}{\partial \lambda_{T\times T}} = -\underset{1\times 1}{p^{-1}}\left(\underset{T\times T}{\mathbf{W}}\,\underset{T\times T}{\mathbf{A}^{'}} + \underset{T\times T}{\mathbf{A}}\,\underset{T\times T}{\mathbf{W}^{'}}\right)$$

b. Turunan pertama $\ln|\mathbf{\Theta}|$ terhadap λ

$$\underset{T\times T}{\mathbf{\Theta}} = \underset{1\times 1}{p^{-1}}\left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)^{'} = \underset{1\times 1}{p^{-1}}\,\underset{T\times T}{\mathbf{A}}\,\underset{T\times T}{\mathbf{A}^{'}}$$

$$\partial \ln|\mathbf{X}| = tr\left(\mathbf{X}^{-1}\partial \mathbf{X}\right)$$

$$\frac{\partial \ln|\mathbf{\Theta}|}{\partial \lambda} = tr\left(\left(\underset{1\times 1}{p^{-1}}\,\underset{T\times T}{\mathbf{A}}\,\underset{T\times T}{\mathbf{A}^{'}}\right)^{-1}\frac{\partial \mathbf{\Theta}}{\partial \lambda}\right)$$

$$\frac{\partial \ln|\mathbf{\Theta}|}{\partial \lambda} = tr\left(\underset{1\times 1}{p}\left(\underset{T\times T}{\mathbf{A}}\,\underset{T\times T}{\mathbf{A}^{'}}\right)^{-1}\left(\underset{1\times 1}{p^{-1}}\left(\underset{T\times T}{\mathbf{W}}\,\underset{T\times T}{\mathbf{A}^{'}} + \underset{T\times T}{\mathbf{A}}\,\underset{T\times T}{\mathbf{W}^{'}}\right)\right)\right)$$

$$\frac{\partial \ln|\mathbf{\Theta}|}{\partial \lambda} = -tr\left(\left(\underset{T\times T}{\mathbf{A}}\,\underset{T\times T}{\mathbf{A}^{'}}\right)^{-1}\underset{T\times T}{\mathbf{W}}\,\underset{T\times T}{\mathbf{A}^{'}}\right) - tr\left(\left(\underset{T\times T}{\mathbf{A}}\,\underset{T\times T}{\mathbf{A}^{'}}\right)^{-1}\underset{T\times T}{\mathbf{A}}\,\underset{T\times T}{\mathbf{W}^{'}}\right)$$

$$\frac{\partial \ln|\mathbf{\Theta}|}{\partial \lambda} = -tr\left(\underset{T\times T}{\mathbf{A}^{'}}\left(\underset{T\times T}{\mathbf{A}^{'}}\right)^{-1}\left(\underset{T\times T}{\mathbf{A}^{-1}}\right)\underset{T\times T}{\mathbf{W}}\right) - tr\left(\left(\underset{T\times T}{\mathbf{A}^{'}}\right)^{-1}\underset{T\times T}{\mathbf{A}^{-1}}\,\underset{T\times T}{\mathbf{A}}\,\underset{T\times T}{\mathbf{W}^{'}}\right)$$

$$\frac{\partial \ln|\mathbf{\Theta}|}{\partial \lambda} = -tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\,\underset{T\times T}{\mathbf{W}}\right) - tr\left(\left(\underset{T\times T}{\mathbf{A}^{'}}\right)^{-1}\underset{T\times T}{\mathbf{W}^{'}}\right)$$

$$\frac{\partial \ln|\mathbf{\Theta}|}{\partial \lambda} = -tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\,\underset{T\times T}{\mathbf{W}}\right) - tr\left(\underset{T\times T}{\mathbf{W}^{'}}\left(\underset{T\times T}{\mathbf{A}^{'}}\right)^{-1}\right)$$

$$\frac{\partial \ln|\mathbf{\Theta}|}{\partial \lambda} = -tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\,\underset{T\times T}{\mathbf{W}}\right) - tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\,\underset{T\times T}{\mathbf{W}}\right)^{'}$$

$$\frac{\partial \ln|\mathbf{\Theta}|}{\partial \lambda} = -2tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\,\underset{T\times T}{\mathbf{W}}\right)$$

c. Turunan pertama $\ln\left|\underset{T\times T}{\mathbf{A}}\right|$ terhadap λ

$$\partial\left(\ln|\mathbf{A}|\right) = tr\left(\mathbf{A}^{-1}\partial \mathbf{A}\right)$$

$$\underset{T\times T}{\mathbf{A}} = \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)$$

$$\frac{\partial \ln\left|\underset{T\times T}{\mathbf{A}}\right|}{\partial \lambda} = tr\left(\left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)^{-1}\frac{\partial\left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}}\right)}{\partial \lambda}\right)$$

$$\frac{\partial \ln \left| \underset{T\times T}{\mathbf{A}} \right|}{\partial \lambda} = tr\left(\left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}} \right)^{-1} \left(-\underset{T\times T}{\mathbf{W}} \right) \right)$$

$$\frac{\partial \ln \left| \underset{T\times T}{\mathbf{A}} \right|}{\partial \lambda} = -tr\left(\left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}} \right)^{-1} \underset{T\times T}{\mathbf{W}} \right)$$

$$\frac{\partial \ln \left| \underset{T\times T}{\mathbf{A}} \right|}{\partial \lambda} = -tr\left(\underset{T\times T}{\mathbf{A}^{-1}} \underset{T\times T}{\mathbf{W}} \right)$$

d. Turunan pertama $\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}$ terhadap λ

$$\underset{T\times T}{\mathbf{A}} = \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}} \right)$$

$$\partial\left(\mathbf{X}^{-1} \right) = -\mathbf{X}^{-1} \left(\partial \mathbf{X} \right) \mathbf{X}^{-1}$$

$$\frac{\partial \underset{T\times T}{\mathbf{A}^{-1}}}{\partial \lambda} = \frac{\partial \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}} \right)^{-1}}{\partial \lambda} = -\left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}} \right)^{-1} \frac{\partial \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}} \right)}{\partial \lambda} \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}} \right)^{-1}$$

$$\frac{\partial \underset{T\times T}{\mathbf{A}^{-1}}}{\partial \lambda} = -\left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}} \right)^{-1} \left(-\underset{T\times T}{\mathbf{W}} \right) \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}} \right)^{-1}$$

$$\underset{T\times T}{\frac{\partial \underset{T\times T}{\mathbf{A}^{-1}}}{\partial \lambda}} = \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}} \right)^{-1} \underset{T\times T}{\mathbf{W}} \left(\underset{T\times T}{\mathbf{I}} - \lambda \underset{T\times T}{\mathbf{W}} \right)^{-1}$$

$$\underset{T\times T}{\frac{\partial \underset{T\times T}{\mathbf{A}^{-1}}}{\partial \lambda}} = \underset{T\times T}{\mathbf{A}^{-1}} \underset{T\times T}{\mathbf{W}} \underset{T\times T}{\mathbf{A}^{-1}}$$

$$\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon} = \left(\underset{T\times T}{\mathbf{A}} \underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\mathbf{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}} \right)' \left(\underset{1\times 1}{p^{-1}} \underset{T\times T}{\mathbf{A}} \underset{T\times T}{\mathbf{A}'} \right)^{-1} \left(\underset{T\times T}{\mathbf{A}} \underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\mathbf{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}} \right)$$

$$\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon} = \underset{1\times 1}{p} \left(\underset{T\times T}{\mathbf{A}^{-1}} \left(\underset{T\times T}{\mathbf{A}} \underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\mathbf{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}} \right) \right)' \underset{T\times T}{\mathbf{A}^{-1}} \left(\underset{T\times T}{\mathbf{A}} \underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\mathbf{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}} \right)$$

$$\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}_{1\times 1} = \underset{1\times 1}{p} \left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}} \underset{T\times(p+1)}{\mathbf{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}} \right)' \left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}} \underset{T\times(p+1)}{\mathbf{K}} \underset{(p+1)\times 1}{\boldsymbol{\beta}} \right)$$

$$\frac{\partial \mathbf{A}_{(x)}^{T} \mathbf{A}_{(x)}}{\partial x} = 2\left(\frac{\partial \mathbf{A}_{(x)}}{\partial x} \right)^{T} \mathbf{A}_{(x)}$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)}{\partial\lambda} = 2\underset{1\times1}{p}\left(\frac{\partial\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)}{\partial\lambda}\right)'\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)}{\partial\lambda} = -2\underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

2. Turunan pertama fungsi *ln likelihood* untuk $L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)$ terhadap λ

$$L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l) = -\frac{1}{2}\ln|\Theta| + \ln|\mathbf{A}| - \frac{1}{2}\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)$$

$$\frac{\partial L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\lambda} = -\frac{1}{2}\left(-2tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\right)\right) - tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\right) + \underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

$$\frac{\partial L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\lambda} = tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\right) - tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\right) + \underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

$$\underset{1\times1}{\frac{\partial L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\lambda}} = \underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

3. Turunan pertama fungsi *ln likelihood* untuk $L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)$ terhadap $\boldsymbol{\beta}$

$$L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l}) = -\frac{1}{2}\ln|\Theta| + \ln|\mathbf{A}| - \frac{1}{2}\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)$$

$$\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon} = \left(\underset{T\times T}{\mathbf{A}}\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\left(\underset{1\times1}{p^{-1}}\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{A}'}\right)^{-1}\left(\underset{T\times T}{\mathbf{A}}\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

$$\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}_{1\times1} = \underset{1\times1}{p}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

$$\frac{\partial}{\partial\mathbf{s}}(\mathbf{x}-\mathbf{As})^T(\mathbf{x}-\mathbf{As}) = -2\mathbf{A}^T(\mathbf{x}-\mathbf{As})$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)}{\partial\boldsymbol{\beta}} = \underset{1\times1}{p}\left[-2\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right]$$

$$\frac{\partial L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}} = -\frac{1}{2}\frac{\partial\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)}{\partial\boldsymbol{\beta}}$$

$$\frac{\partial L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}} = -\frac{1}{2}\left(-2\underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right)$$

$$\underset{(p+1)\times1}{\frac{\partial L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}}} = \underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

4. Turunan kedua $\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\lambda^2}$

$$\frac{\partial\left(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}\right)}{\partial\lambda} = \frac{\partial\left(\mathbf{A}\mathbf{W}^{-1}\right)}{\partial\lambda}\mathbf{A} + \mathbf{A}\mathbf{W}^{-1}\frac{\partial\mathbf{A}}{\partial\lambda}$$

$$\frac{\partial\left(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}\right)}{\partial\lambda} = \frac{\partial(\mathbf{A})\mathbf{W}^{-1}}{\partial\lambda}\mathbf{A} + \mathbf{A}\mathbf{W}^{-1}(-\mathbf{W})$$

$$\frac{\partial\left(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}\right)}{\partial\lambda} = (-\mathbf{W})\mathbf{W}^{-1}\mathbf{A} + \mathbf{A}\mathbf{W}^{-1}(-\mathbf{W})$$

$$\frac{\partial\left(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}\right)}{\partial\lambda} = -\mathbf{A} - \mathbf{A} = -2\mathbf{A}$$

$$\frac{\partial\left(\mathbf{A}^{-1}\mathbf{W}\mathbf{A}^{-1}\right)}{\partial\lambda} = \frac{\partial\left(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}\right)^{-1}}{\partial\lambda} = -\left(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}\right)^{-1}\frac{\partial\left(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}\right)}{\partial\lambda}\left(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}\right)^{-1}$$

$$\frac{\partial\left(\mathbf{A}^{-1}\mathbf{W}\mathbf{A}^{-1}\right)}{\partial\lambda} = -\left(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}\right)^{-1}(-2\mathbf{A})\left(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}\right)^{-1}$$

$$\frac{\partial\left(\mathbf{A}^{-1}\mathbf{W}\mathbf{A}^{-1}\right)}{\partial\lambda} = 2\left(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}\left(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}\right)^{-1}$$

$$\frac{\partial\left(\mathbf{A}^{-1}\mathbf{W}\mathbf{A}^{-1}\right)}{\partial\lambda} = 2\left(\mathbf{A}^{-1}\mathbf{W}\mathbf{A}^{-1}\right)\mathbf{A}\left(\mathbf{A}^{-1}\mathbf{W}\mathbf{A}^{-1}\right)$$

$$\underset{1\times1}{\frac{\partial L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\lambda}} = \underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times1}{l} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)$$

$$\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial \lambda^2} = \underset{1\times1}{p}\left(\frac{\partial\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'}{\partial\lambda}\left(\underset{T\times1}{l}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right.$$

$$\left.+\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\frac{\partial\left(\underset{T\times1}{l}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)}{\partial\lambda}\right)$$

$$\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial \lambda^2} = \underset{1\times1}{p}\left(\left(\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\frac{\partial\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)'}{\partial\lambda}\left(\underset{T\times1}{l}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)+\right.$$

$$\left.+\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\frac{-\partial\left(\underset{T\times T}{\mathbf{A}^{-1}}\right)}{\partial\lambda}\left(\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right)$$

$$\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial \lambda^2} = \underset{1\times1}{p}\left(\left(\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(2\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\underset{T\times T}{\mathbf{A}}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\right)'\left(\underset{T\times1}{l}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right.$$

$$\left.-\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\left(\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right)$$

$$\underset{1\times1}{\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial \lambda^2}} = \underset{1\times1}{p}\left(2\left(\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\underset{T\times T}{\mathbf{A}^{-1}}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\right)'\left(\underset{T\times1}{l}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right.$$

$$\left.-\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\left(\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right)$$

5. Turunan kedua $\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'}$

$$\underset{(p+1)\times1}{\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}}} = \underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)\left(\underset{T\times1}{l}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)$$

$$\underset{(p+1)\times1}{\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}}} = \underset{1\times1}{p}\left[\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\underset{T\times1}{l}-\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right]$$

$$\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'} = \underset{1\times1}{p}\left(-\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\frac{\partial\boldsymbol{\beta}}{\partial\boldsymbol{\beta}'}\right)$$

$$\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'} = -\underset{1\times1}{p}\left(\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times(p+1)}{\mathbf{I}}\right)$$

$$\underset{(p+1)\times(p+1)}{\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'}} = -\underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}$$

6. Turunan kedua $\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\lambda\partial\boldsymbol{\beta}'}$

$$\underset{1\times1}{\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\lambda\partial\boldsymbol{\beta}'}} = \underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times1}{l}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)$$

$$\frac{\partial}{\partial\mathbf{x}}\mathbf{s}'\mathbf{A}\mathbf{r} = \left[\frac{\partial\mathbf{s}}{\partial\mathbf{x}}\right]'\mathbf{A}\mathbf{r} + \left[\frac{\partial\mathbf{r}}{\partial\mathbf{x}}\right]'\mathbf{A}'\mathbf{s}$$

$$\frac{\partial\mathbf{A}_{(x)}^T\mathbf{B}_{(x)}}{\partial x} = \left(\frac{\partial\mathbf{A}_{(x)}}{\partial x}\right)^T\mathbf{B}_{(x)} + \left(\frac{\partial\mathbf{B}_{(x)}}{\partial x}\right)^T\mathbf{A}_{(x)}$$

$$\frac{\partial\mathbf{A}_{(x)}^T\mathbf{B}_{(x)}}{\partial x^T} = \left(\frac{\partial\mathbf{B}_{(x)}\mathbf{A}_{(x)}}{\partial x}\right)^T = \left[\left(\frac{\partial\mathbf{B}_{(x)}}{\partial x}\right)^T\mathbf{A}_{(x)} + \left(\frac{\partial\mathbf{A}_{(x)}}{\partial x}\right)^T\mathbf{B}_{(x)}\right]^T = \mathbf{A}_{(x)}^T\frac{\partial\mathbf{B}_{(x)}}{\partial x} + \mathbf{B}_{(x)}^T\frac{\partial\mathbf{A}_{(x)}}{\partial x}$$

$$\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\lambda\partial\boldsymbol{\beta}'} = \underset{1\times1}{p}\left(\left(\underset{T\times1}{l}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\frac{\partial\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)}{\partial\boldsymbol{\beta}'} + \right.$$

$$\left.\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\frac{\partial\left(\underset{T\times1}{l}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)}{\partial\boldsymbol{\beta}'}\right)$$

$$\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\lambda\partial\boldsymbol{\beta}'} = \underset{1\times1}{p}\left(\left(\underset{T\times1}{l}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\frac{\partial\boldsymbol{\beta}}{\partial\boldsymbol{\beta}'}\right) + \right.$$

$$\left.\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\frac{\partial\boldsymbol{\beta}}{\partial\boldsymbol{\beta}'}\right)\right)$$

$$\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\lambda\partial\boldsymbol{\beta}'} = \underset{1\times1}{p}\left(\left(\underset{T\times1}{l}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times(p+1)}{\mathbf{I}}\right) + \right.$$

$$\left.\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times(p+1)}{\mathbf{I}}\right)\right)$$

$$\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\lambda\partial\boldsymbol{\beta}'} = \underset{1\times1}{p}\left(\underset{T\times1}{l} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)$$

$$-\underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times1}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)$$

7. Turunan kedua $\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}\partial\lambda}$

$$\underset{(p+1)\times1}{\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}}} = \underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)\left(\underset{T\times1}{l} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)$$

$$\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}\partial\lambda} = \underset{1\times1}{p}\left[\frac{\partial\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'}{\partial\lambda}\left(\underset{T\times1}{l} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right) + \left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\frac{\partial\left(\underset{T\times1}{l} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)}{\partial\lambda}\right]$$

$$\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}\partial\lambda} = \underset{1\times1}{p}\left[\left(\frac{\partial\underset{T\times T}{\mathbf{A}^{-1}}}{\partial\lambda}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times1}{l} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right) + \left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(-\frac{\partial\underset{T\times T}{\mathbf{A}^{-1}}}{\partial\lambda}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right]$$

$$\underset{T\times T}{\frac{\partial\underset{T\times T}{\mathbf{A}^{-1}}}{\partial\lambda}} = \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}$$

$$\underset{(p+1)\times1}{\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}\partial\lambda}} = \underset{1\times1}{p}\left(\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times1}{l} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right) + \right.$$

$$\left. -\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right)$$

1) Elemen matriks informasi

   a. Elemen matriks (1,1) $\tilde{\boldsymbol{\psi}}_{\lambda\lambda}$

$$\underset{1\times1}{\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}\partial\lambda}} = \underset{1\times1}{p}\left(2\left(\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\underset{T\times T}{\mathbf{A}^{-1}}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\right)'\left(\underset{T\times1}{l} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right) + \right.$$

$$\left. -\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\left(\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right)$$

$$\tilde{\boldsymbol{\psi}}_{\lambda\lambda} = E\left(-\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\lambda^2}\right)$$

$$\tilde{\boldsymbol{\psi}}_{\lambda\lambda} = E\left(\begin{matrix} -\underset{1\times1}{p}\left(2\left(\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\underset{T\times T}{\mathbf{A}^{-1}}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\right)'\left(\underset{T\times1}{l} - \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right) + \right. \\ \left. -\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\left(\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right) \end{matrix}\right)$$

$$\tilde{\boldsymbol{\psi}}_{\lambda\lambda}=-\underset{1\times1}{p}\left(2\left(\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\underset{T\times T}{\mathbf{A}^{-1}}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\right)'\left(E\left(\underset{T\times1}{l}\right)-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)+\right.$$

$$\left.-\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\left(\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right)$$

$$\tilde{\boldsymbol{\psi}}_{\lambda\lambda}=\underset{1\times1}{p}\left(\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\left(\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)+\right.$$

$$\left.-2\left(\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\underset{T\times T}{\mathbf{A}^{-1}}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\right)'\left(\underset{T\times1}{e}\underset{1\times1}{\eta}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right)$$

$$\underset{1\times1}{\tilde{\boldsymbol{\psi}}_{\lambda\lambda}}=\underset{1\times1}{p}\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)-2\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{W}'}\left(\underset{T\times1}{e}\underset{1\times1}{\eta}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right]$$

b. Elemen matriks (2,2) $\tilde{\psi}_{\beta\beta}$

$$\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'}_{(p+1)\times(p+1)}=-\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}$$

$$\tilde{\boldsymbol{\psi}}_{\beta\beta}=E\left(-\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'}\right)$$

$$\tilde{\boldsymbol{\psi}}_{\beta\beta}=E\left(-\left(\underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)\right)$$

$$\underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\beta\beta}}=\underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}$$

Pada saat $\lambda=0$

$$\underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\beta\beta}}=\underset{1\times1}{p}\left(\underset{(p+1)\times T}{\mathbf{K}'}\underset{T\times(p+1)}{\mathbf{K}}\right)$$

c. Elemen matriks (1,2) $\tilde{\psi}_{\lambda\beta}$

$$\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\lambda\partial\boldsymbol{\beta}'}_{1\times(p+1)}=\underset{1\times1}{p}\left[\left(\underset{T\times1}{l}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)-\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)\right]$$

$$\hat{\boldsymbol{\psi}}_{\lambda\boldsymbol{\beta}}=E\left(-\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\lambda\partial\boldsymbol{\beta}}\right)$$

$$\hat{\boldsymbol{\psi}}_{\lambda\boldsymbol{\beta}}=E\left[-\underset{1\times1}{p}\left(\left(\underset{T\times1}{l}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)\right)+\left(-\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)\right)\right]$$

$$\hat{\boldsymbol{\psi}}_{\lambda\boldsymbol{\beta}}=\underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)+\left(-\left(E\left(\underset{T\times1}{l}\right)-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)\right)$$

$$\underset{1\times(p+1)}{\hat{\boldsymbol{\psi}}_{\lambda\boldsymbol{\beta}}}=\underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)+\left(\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)\right)$$

pada saat λ=0

$$\underset{1\times(p+1)}{\hat{\boldsymbol{\psi}}_{\lambda\boldsymbol{\beta}}}=\underset{1\times1}{p}\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\underset{T\times(p+1)}{\mathbf{K}}-\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\right)\right]$$

d. Elemen matriks (1,2) $\tilde{\boldsymbol{\psi}}_{\lambda\boldsymbol{\beta}}$

$$\underset{(p+1)\times1}{\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}\partial\lambda}}=\underset{1\times1}{p}\left[\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times1}{l}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)+\left(-\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right)\right]$$

$$\hat{\boldsymbol{\psi}}_{\boldsymbol{\beta}\lambda}=E\left(-\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\boldsymbol{\beta}\partial\lambda}\right)$$

$$\hat{\boldsymbol{\psi}}_{\boldsymbol{\beta}\lambda}=E\left[-\underset{1\times1}{p}\left(\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times1}{l}-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right)+\left(-\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right)\right]$$

$$\hat{\boldsymbol{\psi}}_{\boldsymbol{\beta}\lambda}=\underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)+\left(-\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(E\left(\underset{T\times1}{l}\right)-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right)$$

$$\underset{(p+1)\times1}{\hat{\boldsymbol{\psi}}_{\boldsymbol{\beta}\lambda}}=\underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)+\left(\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right)$$

pada saat λ=0

$$\underset{(p+1)\times1}{\hat{\boldsymbol{\psi}}_{\boldsymbol{\beta}\lambda}}=\underset{1\times1}{p}\left[\underset{T\times(p+1)}{\mathbf{K}}{}'\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)-\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right]$$

2. Matriks Informasi

Matriks informasi pada saat λ=0 adalah

$$\tilde{\boldsymbol{\psi}}_{\theta}=\begin{pmatrix}\underset{1\times1}{\tilde{\boldsymbol{\psi}}_{\lambda\lambda}} & \underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\lambda\boldsymbol{\beta}}}\\ \underset{(p+1)\times1}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\lambda}} & \underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}'}}\end{pmatrix}\text{ dimana}$$

$$\underset{1\times1}{\tilde{\boldsymbol{\psi}}_{\lambda\lambda}}=\underset{1\times1}{p}\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)-2\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right]$$

$$\underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\lambda\boldsymbol{\beta}}}=\underset{1\times1}{p}\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\underset{T\times(p+1)}{\mathbf{K}}-\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\right)\right]$$

$$\underset{(p+1)\times1}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\lambda}}=\underset{1\times1}{p}\left[\underset{T\times(p+1)}{\mathbf{K}}\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)-\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right]$$

$$\underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}'}}=\underset{1\times1}{p}\left(\underset{T\times(p+1)}{\mathbf{K}}\underset{T\times(p+1)}{\mathbf{K}}\right)$$

3. Invers Matriks Informasi pada saat λ=0

Jika ada matriks partisi $\mathbf{C}=\left(\begin{array}{c|c}\mathbf{C}_1 & \mathbf{C}_2\\ \hline \mathbf{C}_3 & \mathbf{C}_4\end{array}\right)$ maka matriks invers nya adalah $\mathbf{C}^{-1}=\left(\begin{array}{c|c}\mathbf{C}_{E1} & \mathbf{C}_{E2}\\ \hline \mathbf{C}_{E3} & \mathbf{C}_{E4}\end{array}\right)$ dimana

$$\mathbf{C}_{E1}=\left(\mathbf{C}_1-\mathbf{C}_2\mathbf{C}_4^{-1}\mathbf{C}_3\right)^{-1}$$
$$\mathbf{C}_{E2}=\left(-\mathbf{C}_{E1}\mathbf{C}_2\mathbf{C}_4^{-1}\right)$$
$$\mathbf{C}_{E3}=\left(-\mathbf{C}_4^{-1}\mathbf{C}_3\mathbf{C}_{E1}\right)$$
$$\mathbf{C}_{E4}=\left(\mathbf{C}_4^{-1}-\mathbf{C}_4^{-1}\mathbf{C}_3\mathbf{C}_{E2}\right)$$

Elemen baris pertama dan kolom pertama dari invers matriks informasi

$$\tilde{\boldsymbol{\psi}}_{\theta}=\begin{pmatrix}\underset{1\times1}{\tilde{\boldsymbol{\psi}}_{\lambda\lambda}} & \underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\lambda\boldsymbol{\beta}}}\\ \underset{(p+1)\times1}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\lambda}} & \underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}'}}\end{pmatrix}$$ adalah

$$\mathbf{C}_{E1}=\left(\mathbf{C}_1-\mathbf{C}_2\mathbf{C}_4^{-1}\mathbf{C}_3\right)^{-1}$$

$$\underset{1\times1}{\tilde{\boldsymbol{\psi}}_{\lambda\lambda}^{-1}}=\left(\underset{1\times1}{\tilde{\boldsymbol{\psi}}_{\lambda\lambda}}-\underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\lambda\boldsymbol{\beta}}}\left(\underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}'}}\right)^{-1}\underset{(p+1)\times1}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\lambda}}\right)^{-1}$$

$$\underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\lambda\boldsymbol{\beta}}}\left(\underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}'}}\right)^{-1}\underset{(p+1)\times1}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\lambda}}=\underset{1\times1}{p}\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\underset{T\times(p+1)}{\mathbf{K}}-\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\right)\right]\times$$
$$\left(\underset{1\times1}{p}\underset{T\times(p+1)}{\mathbf{K}'}\underset{T\times(p+1)}{\mathbf{K}}\right)^{-1}\times\underset{1\times1}{p}\left[\underset{T\times(p+1)}{\mathbf{K}'}\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}-\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right]$$

$$\underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\lambda\boldsymbol{\beta}}}\left(\underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}'}}\right)^{-1}\underset{(p+1)\times1}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\lambda}}=\underset{1\times1}{p}\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\underset{T\times(p+1)}{\mathbf{K}}-\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\right)\right]\times$$
$$\underset{T\times(p+1)}{\mathbf{K}'}\left(\underset{T\times(p+1)}{\mathbf{K}}\right)^{-1}\left[\underset{T\times(p+1)}{\mathbf{K}'}\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}-\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\right)'\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right]$$

$$\underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\lambda\boldsymbol{\beta}}}\left(\underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}'}}\right)^{-1}\underset{(p+1)\times1}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\lambda}}=\underset{1\times1}{p}\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'-\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{W}}\right]\times$$
$$\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)-\underset{T\times T}{\mathbf{W}'}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right]$$

$$\underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\lambda\boldsymbol{\beta}}}\left(\underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}'}}\right)^{-1}\underset{(p+1)\times1}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\lambda}}=\underset{1\times1}{p}\left(\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)-\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{W}}\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)+\right.$$

$$-\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)+\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'$$

$$\left.\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{W}'}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right)$$

$$\underset{1\times1}{\tilde{\boldsymbol{\psi}}_{\lambda\lambda}}=\underset{1\times1}{p}\left[\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)-2\left(\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{W}'}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\boldsymbol{\eta}}-\underset{T\times(p+1)}{\mathbf{K}}\underset{(p+1)\times T}{\boldsymbol{\beta}}\right)\right]$$

Lampiran 6. Turunan Pertama dan Kedua Fungsi *ln likelihood* $L(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};l)$ pada Model SERM-SEM

1. Turunan pertama fungsi *ln likelihood* untuk $\mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})$ terhadap ρ

   a. Turunan pertama $\boldsymbol{\Theta}$ terhadap ρ

   karena hasil perkalian $\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{W}}\right)^{'}$ adalah matriks simetris

$$\underset{T\times T}{\boldsymbol{\Theta}}=\underset{T\times T}{\mathbf{A}}\;\underset{1\times 1}{p^{-1}}\;\underset{T\times T}{\mathbf{A}^{'}}=\underset{1\times 1}{p^{-1}}\;\underset{T\times T}{\mathbf{A}}\;\underset{T\times T}{\mathbf{A}^{'}}$$

$$\underset{T\times T}{\boldsymbol{\Theta}}=\underset{1\times 1}{p^{-1}}\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{W}}\right)^{'}$$

$$\frac{\partial \mathbf{A}_{(x)}^{T}\mathbf{A}_{(x)}}{\partial x}=2\left(\frac{\partial \mathbf{A}_{(x)}}{\partial x}\right)^{T}\mathbf{A}_{(x)}$$

$$\frac{\partial\boldsymbol{\Theta}}{\partial\rho}=\frac{\partial\left(\underset{1\times 1}{p^{-1}}\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{W}}\right)^{'}\right)}{\partial\rho}=\underset{1\times 1}{p^{-1}}\frac{\partial\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{W}}\right)^{'}}{\partial\rho}$$

$$\frac{\partial\boldsymbol{\Theta}}{\partial\rho}=2\underset{1\times 1}{p^{-1}}\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{W}}\right)\frac{\partial\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{W}}\right)^{'}}{\partial\rho}$$

$$\frac{\partial\boldsymbol{\Theta}}{\partial\rho}=2\underset{1\times 1}{p^{-1}}\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{W}}\right)\left(-\underset{T\times T}{\mathbf{W}}\right)^{'}$$

$$\frac{\partial\boldsymbol{\Theta}}{\partial\rho}=-2\underset{1\times 1}{p^{-1}}\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{W}}\right)\underset{T\times T}{\mathbf{W}^{'}}$$

$$\frac{\partial\boldsymbol{\Theta}}{\partial\rho}=-2\underset{1\times 1}{p^{-1}}\;\underset{T\times T}{\mathbf{A}}\;\underset{T\times T}{\mathbf{W}^{'}}$$

   b. Turunan pertama $\ln|\boldsymbol{\Theta}|$ terhadap ρ

$$\underset{T\times T}{\boldsymbol{\Theta}}=\underset{1\times 1}{p^{-1}}\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times T}{\mathbf{I}}-\rho\underset{T\times T}{\mathbf{W}}\right)^{'}=\underset{1\times 1}{p^{-1}}\;\underset{T\times T}{\mathbf{A}}\;\underset{T\times T}{\mathbf{A}^{'}}$$

$$\partial\ln|\mathbf{X}|=Tr\left(\mathbf{X}^{-1}\partial\mathbf{X}\right)$$

$$\frac{\partial\boldsymbol{\Theta}}{\partial\rho}=-2\underset{1\times 1}{p^{-1}}\;\underset{T\times T}{\mathbf{A}}\;\underset{T\times T}{\mathbf{W}^{'}}$$

$$\frac{\partial \ln|\boldsymbol{\Theta}|}{\partial \rho} = tr\left(\left(\underset{1\times1}{p^{-1}}\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{A}'}\right)^{-1}\frac{\partial \boldsymbol{\Theta}}{\partial \rho}\right)$$

$$\frac{\partial \ln|\boldsymbol{\Theta}|}{\partial \rho} = tr\left(\underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{A}'}\right)^{-1}\left(-2\underset{1\times1}{p^{-1}}\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{W}'}\right)\right)$$

$$\frac{\partial \ln|\boldsymbol{\Theta}|}{\partial \rho} = -2tr\left(\left(\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{A}'}\right)^{-1}\left(\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{W}'}\right)\right)$$

$$\frac{\partial \ln|\boldsymbol{\Theta}|}{\partial \rho} = -2tr\left(\left(\underset{T\times T}{\mathbf{A}'}\right)^{-1}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{W}'}\right)$$

$$\frac{\partial \ln|\boldsymbol{\Theta}|}{\partial \rho} = -2tr\left(\left(\underset{T\times T}{\mathbf{A}'}\right)^{-1}\underset{T\times T}{\mathbf{W}'}\right)$$

$$\frac{\partial \ln|\boldsymbol{\Theta}|}{\partial \rho} = -2tr\left(\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)'$$

Sifat-sifat trace

$$tr(\mathbf{A})' = tr(\mathbf{A})$$

$$tr(\mathbf{AB}) = tr(\mathbf{BA})$$

Sehingga

$$\frac{\partial \ln|\boldsymbol{\Theta}|}{\partial \rho} = -2tr\left(\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)' = -2tr\left(\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right) = -2tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\right)$$

c. Turunan pertama $\ln\left|\underset{T\times T}{\mathbf{A}}\right|$ terhadap ρ

$$\partial\left(\ln|\mathbf{A}|\right) = tr\left(\mathbf{A}^{-1}\partial\mathbf{A}\right)$$

$$\underset{T\times T}{\mathbf{A}} = \left(\underset{T\times T}{\mathbf{I}} - \rho\underset{T\times T}{\mathbf{W}}\right)$$

$$\frac{\partial \ln\left|\underset{T\times T}{\mathbf{A}}\right|}{\partial \rho} = tr\left(\left(\underset{T\times T}{\mathbf{I}} - \rho\underset{T\times T}{\mathbf{W}}\right)^{-1}\frac{\partial\left(\underset{T\times T}{\mathbf{I}} - \rho\underset{T\times T}{\mathbf{W}}\right)}{\partial \rho}\right)$$

$$\frac{\partial \ln\left|\underset{T\times T}{\mathbf{A}}\right|}{\partial \rho} = tr\left(\left(\underset{T\times T}{\mathbf{I}} - \rho\underset{T\times T}{\mathbf{W}}\right)^{-1}\left(-\underset{T\times T}{\mathbf{W}}\right)\right)$$

$$\frac{\partial \ln\left|\underset{T\times T}{\mathbf{A}}\right|}{\partial \rho} = -tr\left(\left(\underset{T\times T}{\mathbf{I}} - \rho\underset{T\times T}{\mathbf{W}}\right)^{-1}\underset{T\times T}{\mathbf{W}}\right)$$

$$\frac{\partial \ln \left|\underset{T\times T}{\mathbf{A}}\right|}{\partial \rho} = -tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\right)$$

d. Turunan pertama $\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}$ terhadap ρ

$$-\frac{1}{2}p\left(\mathbf{A}\boldsymbol{l}-\mathbf{A}\boldsymbol{K}\boldsymbol{\beta}\right)'\left(\mathbf{A}\mathbf{A}'\right)^{-1}\left(\mathbf{A}\boldsymbol{l}-\mathbf{A}\boldsymbol{K}\boldsymbol{\beta}\right)$$

$$\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon} = -\frac{1}{2}\underset{1\times 1}{p}\left(\underset{T\times T}{\mathbf{A}}\underset{T\times 1}{\boldsymbol{l}}-\underset{T\times T}{\mathbf{A}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{A}'}\right)^{-1}\left(\underset{T\times T}{\mathbf{A}}\underset{T\times 1}{\boldsymbol{l}}-\underset{T\times T}{\mathbf{A}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)}{\partial\rho} = -\frac{1}{2}\underset{1\times 1}{p}\left[\frac{\partial\left(\underset{T\times T}{\mathbf{A}}\underset{T\times 1}{\boldsymbol{l}}-\underset{T\times T}{\mathbf{A}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{A}'}\right)^{-1}}{\partial\rho}\times\left(\underset{T\times T}{\mathbf{A}}\underset{T\times 1}{\boldsymbol{l}}-\underset{T\times T}{\mathbf{A}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right]+$$

$$-\frac{1}{2}\underset{1\times 1}{p}\left[\left(\underset{T\times T}{\mathbf{A}}\underset{T\times 1}{\boldsymbol{l}}-\underset{T\times T}{\mathbf{A}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{A}'}\right)^{-1}\times\frac{\partial\left(\underset{T\times T}{\mathbf{A}}\underset{T\times 1}{\boldsymbol{l}}-\underset{T\times T}{\mathbf{A}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)}{\partial\rho}\right]$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)}{\partial\rho} = -\frac{1}{2}\underset{1\times 1}{p}\left[\frac{\partial\left(\underset{T\times 1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}'}\left(\underset{T\times T}{\mathbf{A}'}\right)^{-1}\underset{T\times T}{\mathbf{A}^{-1}}}{\partial\rho}\times\left(\underset{T\times T}{\mathbf{A}}\underset{T\times 1}{\boldsymbol{l}}-\underset{T\times T}{\mathbf{A}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right]+$$

$$-\frac{1}{2}\underset{1\times 1}{p}\left[\left(\underset{T\times 1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}'}\left(\underset{T\times T}{\mathbf{A}'}\right)^{-1}\underset{T\times T}{\mathbf{A}^{-1}}\times\frac{\partial\underset{T\times T}{\mathbf{A}}\left(\underset{T\times 1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)}{\partial\rho}\right]$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)}{\partial\rho} = -\frac{1}{2}\underset{1\times 1}{p}\left[\left(\underset{T\times 1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)'\frac{\partial\underset{T\times T}{\mathbf{A}^{-1}}}{\partial\rho}\times\left(\underset{T\times T}{\mathbf{A}}\underset{T\times 1}{\boldsymbol{l}}-\underset{T\times T}{\mathbf{A}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right]+$$

$$-\frac{1}{2}\underset{1\times 1}{p}\left[\left(\underset{T\times 1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\times\frac{\partial\underset{T\times T}{\mathbf{A}}\left(\underset{T\times 1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)}{\partial\rho}\right]$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)}{\partial\rho}=-\frac{1}{2}\underset{1\times1}{p}\left[\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}^{-1}}\left(-\underset{T\times T}{\mathbf{W}}\right)\underset{T\times T}{\mathbf{A}^{-1}}\right)\underset{T\times T}{\mathbf{A}}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right]+$$

$$-\frac{1}{2}\underset{1\times1}{p}\left[\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\left(-\underset{T\times T}{\mathbf{W}}\right)\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right]$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)}{\partial\rho}=\frac{1}{2}\underset{1\times1}{p}\left[\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right]+$$

$$\frac{1}{2}\underset{1\times1}{p}\left[\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right]$$

$$\underset{1\times1}{\frac{\partial\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)}{\partial\rho}}=\underset{1\times1}{p}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

Turunan pertama fungsi *ln likelihood* untuk $\mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})$ terhadap ρ

$$\frac{\partial\ln|\boldsymbol{\Theta}|}{\partial\rho}=-2tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\right)$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)}{\partial\rho}=\underset{1\times1}{p}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

$$\frac{\partial\ln\left|\underset{T\times T}{\mathbf{A}}\right|}{\partial\rho}=-tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\right)$$

$$\mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};l)=-\frac{1}{2}\ln|\Theta|+\ln|\mathbf{A}|-\frac{1}{2}\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)$$

$$\frac{\partial\mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\rho}=-\frac{1}{2}\left(-2tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\right)\right)-tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\right)+\underset{1\times1}{p}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

$$\frac{\partial\mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\rho}=tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\right)-tr\left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\right)+\underset{1\times1}{p}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

$$\underset{1\times1}{\frac{\partial\mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\rho}}=\underset{1\times1}{p}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

2. Turunan pertama fungsi *ln likelihood* untuk $\mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})$ terhadap $\boldsymbol{\beta}$

$$\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}=-\frac{1}{2}\underset{1\times1}{p}\left(\underset{T\times T}{\mathbf{A}}\underset{T\times1}{\boldsymbol{l}}-\underset{T\times T}{\mathbf{A}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{A}'}\right)^{-1}\left(\underset{T\times T}{\mathbf{A}}\underset{T\times1}{\boldsymbol{l}}-\underset{T\times T}{\mathbf{A}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)}{\partial\boldsymbol{\beta}}=-\frac{1}{2}\underset{1\times1}{p}\left[\frac{\partial\left(\underset{T\times T}{\mathbf{A}}\underset{T\times1}{\boldsymbol{l}}-\underset{T\times T}{\mathbf{A}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{A}'}\right)^{-1}}{\partial\boldsymbol{\beta}}\times\left(\underset{T\times T}{\mathbf{A}}\underset{T\times1}{\boldsymbol{l}}-\underset{T\times T}{\mathbf{A}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right]+$$

$$-\frac{1}{2}\underset{1\times1}{p}\left[\left(\underset{T\times T}{\mathbf{A}}\underset{T\times1}{\boldsymbol{l}}-\underset{T\times T}{\mathbf{A}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\left(\underset{T\times T}{\mathbf{A}}\underset{T\times T}{\mathbf{A}'}\right)^{-1}\times\frac{\partial\left(\underset{T\times T}{\mathbf{A}}\underset{T\times1}{\boldsymbol{l}}-\underset{T\times T}{\mathbf{A}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)}{\partial\boldsymbol{\beta}}\right]$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)}{\partial\boldsymbol{\beta}}=-\frac{1}{2}\underset{1\times1}{p}\left[\frac{\partial\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}'}\left(\underset{T\times T}{\mathbf{A}'}\right)^{-1}\underset{T\times T}{\mathbf{A}^{-1}}}{\partial\boldsymbol{\beta}}\times\underset{T\times T}{\mathbf{A}}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times T}{\mathbf{A}}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right]+$$

$$-\frac{1}{2}\underset{1\times1}{p}\left[\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}'}\left(\underset{T\times T}{\mathbf{A}'}\right)^{-1}\underset{T\times T}{\mathbf{A}^{-1}}\times\underset{T\times T}{\mathbf{A}}\frac{\partial\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)}{\partial\boldsymbol{\beta}}\right]$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)}{\partial\boldsymbol{\beta}}=-\frac{1}{2}\underset{1\times1}{p}\left[\frac{\partial\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'}{\partial\boldsymbol{\beta}}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right]+$$

$$-\frac{1}{2}\underset{1\times1}{p}\left[\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\times\frac{\partial\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)}{\partial\boldsymbol{\beta}}\right]$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)}{\partial\boldsymbol{\beta}}=-\frac{1}{2}\underset{1\times1}{p}\left[\left(-\underset{T\times(p+1)}{\boldsymbol{K}}\right)'\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)+\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\left(-\underset{T\times(p+1)}{\boldsymbol{K}}\right)\right]$$

$$\underset{(p+1)\times1}{\frac{\partial\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)}{\partial\boldsymbol{\beta}}}=\underset{1\times1}{p}\underset{(p+1)\times T}{\boldsymbol{K}'}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

$$\mathcal{L}\left(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};l\right)=-\frac{1}{2}\ln\left|\Theta\right|+\ln\left|\mathbf{A}\right|-\frac{1}{2}\left(\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}\right)$$

$$\frac{\partial\mathcal{L}\left(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l}\right)}{\partial\boldsymbol{\beta}}=\frac{\partial\boldsymbol{\varepsilon}'\boldsymbol{\Theta}^{-1}\boldsymbol{\varepsilon}}{\partial\boldsymbol{\beta}}$$

$$\underset{(p+1)\times1}{\frac{\partial\mathcal{L}\left(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l}\right)}{\partial\boldsymbol{\beta}}}=\underset{1\times1}{p}\underset{(p+1)\times T}{\boldsymbol{K}'}\left(\underset{T\times1}{\boldsymbol{l}}-\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

3. Turunan kedua $\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \rho^2}$

$$\underset{1\times 1}{\frac{\partial \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \rho}} = \underset{1\times 1}{p}\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)' \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)$$

$$\underset{1\times 1}{\frac{\partial \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \rho}} = \underset{1\times 1}{p}\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)' \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)$$

$$\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \rho^2} = \underset{1\times 1}{p}\left[\frac{\partial\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)' \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}}{\partial \rho} \times \left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right]$$

$$+ \underset{1\times 1}{p}\left[\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)' \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}} \frac{\partial\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)'}{\partial \rho}\right]$$

$$\underset{1\times 1}{\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \rho^2}} = -\underset{1\times 1}{p}\left[\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)' \underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right]$$

$$\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \rho^2} = \underset{1\times 1}{p}\left[\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)' \frac{\partial \underset{T\times T}{\mathbf{A}^{-1}}}{\partial \rho}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right]$$

$$\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \rho^2} = -\underset{1\times 1}{p}\left[\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)' \left(-\underset{T\times T}{\mathbf{A}^{-1}}\frac{\partial \underset{T\times T}{(\mathbf{I}-\rho\mathbf{W})}}{\partial \rho}\underset{T\times T}{\mathbf{A}^{-1}}\right)\underset{T\times T}{\mathbf{W}}\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right]$$

$$\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \rho^2} = -\underset{1\times 1}{p}\left[\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)' \left(-\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{(-\mathbf{W})}\underset{T\times T}{\mathbf{A}^{-1}}\right)\underset{T\times T}{\mathbf{W}}\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right]$$

$$\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \rho^2} = -\underset{1\times 1}{p}\left[\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)' \left(\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\right)\underset{T\times T}{\mathbf{W}}\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)\right]$$

4. Turunan kedua $\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}'}$

$$\underset{(p+1)\times 1}{\frac{\partial \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial \boldsymbol{\beta}}} = \underset{1\times 1}{p}\underset{(p+1)\times T}{\boldsymbol{K}'}\left(\underset{T\times 1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right) = \left(\underset{1\times 1}{p}\underset{(p+1)\times T}{\boldsymbol{K}'}\underset{T\times 1}{\boldsymbol{l}} - \underset{1\times 1}{p}\underset{(p+1)\times T}{\boldsymbol{K}'}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times 1}{\boldsymbol{\beta}}\right)$$

$$\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'} = \underset{1\times1}{p}\frac{\partial\left(\underset{(p+1)\times T}{\boldsymbol{K}'}\underset{T\times1}{\boldsymbol{l}}\right)}{\partial\boldsymbol{\beta}} - \underset{1\times1}{p}\frac{\partial\left(\underset{(p+1)\times T}{\boldsymbol{K}'}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)}{\partial\boldsymbol{\beta}}$$

$$\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'} = -\underset{1\times1}{p}\underset{(p+1)\times T}{\boldsymbol{K}'}\underset{T\times(p+1)}{\boldsymbol{K}}\frac{\partial\left(\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)}{\partial\boldsymbol{\beta}}$$

$$\underset{(p+1)\times(p+1)}{\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'}} = -\underset{1\times1}{p}\underset{(p+1)\times T}{\boldsymbol{K}'}\underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times(p+1)}{\mathbf{I}} = -\underset{1\times1}{p}\underset{(p+1)\times T}{\boldsymbol{K}'}\underset{T\times(p+1)}{\boldsymbol{K}}$$

5. Turunan kedua $\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\rho\partial\boldsymbol{\beta}'}$

$$\underset{1\times1}{\frac{\partial \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\rho}} = \underset{1\times1}{p}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

$$\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\rho\partial\boldsymbol{\beta}'} = \underset{1\times1}{p}\left[\frac{\partial\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}}{\partial\boldsymbol{\beta}'}\times\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right] +$$

$$\underset{1\times1}{p}\left[\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\times\frac{\partial\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)}{\partial\boldsymbol{\beta}'}\right]$$

$$\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\rho\partial\boldsymbol{\beta}'} = \underset{1\times1}{p}\left[\left(\frac{\partial\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)}{\partial\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right] +$$

$$\underset{1\times1}{p}\left[\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\times\frac{\partial\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)}{\partial\boldsymbol{\beta}'}\left(\frac{\partial\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'}{\partial\boldsymbol{\beta}}\right)'\right]$$

$$\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\rho\partial\boldsymbol{\beta}'} = \underset{1\times1}{p}\left[\left(-\underset{(p+1)\times T}{\boldsymbol{K}'}\right)\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right] +$$

$$\underset{1\times1}{p}\left[\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\times\left(\left(-\underset{T\times(p+1)}{\boldsymbol{K}}\right)'\right)'\right]$$

$$\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\rho\partial\boldsymbol{\beta}'} = -\underset{1\times1}{p}\left[\underset{(p+1)\times T}{\boldsymbol{K}'}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right] -$$

$$\underset{1\times1}{p}\left[\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}\right]$$

$$\underset{1\times(p+1)}{\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\rho\partial\boldsymbol{\beta}'}} = -2\underset{1\times1}{p}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}$$

6. Turunan kedua $\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}\partial\rho}$

$$\underset{(p+1)\times1}{\frac{\partial \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}}} = \underset{1\times1}{p}\underset{(p+1)\times T}{\boldsymbol{K}'}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

$$\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}\partial\rho} = \underset{1\times1}{p}\frac{\partial\underset{(p+1)\times T}{\boldsymbol{K}'}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)}{\partial\rho}$$

$$\underset{(p\times1)\times1}{\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}\partial\rho}} = \mathbf{0}$$

2) Elemen matriks informasi

e. Elemen matriks (1,1) $\tilde{\boldsymbol{\psi}}_{\rho\rho}$

$$\underset{1\times1}{\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\rho^2}} = -\underset{1\times1}{p}\left[\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right]$$

$$\underset{1\times1}{\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\rho^2}} = -\underset{1\times1}{p}\left[\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right]$$

$$\tilde{\boldsymbol{\psi}}_{\rho\rho} = E\left(-\frac{\partial^2 L(\lambda,\boldsymbol{\beta},\boldsymbol{\Theta};l)}{\partial\rho^2}\right)$$

$$\tilde{\boldsymbol{\psi}}_{\rho\rho} = E\left(-\left(-\underset{1\times1}{p}\left[\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)^{'}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right]\right)\right)$$

$$\tilde{\boldsymbol{\psi}}_{\rho\rho} = E\left(\underset{1\times1}{p}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)^{'}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right)$$

$$\tilde{\boldsymbol{\psi}}_{\rho\rho} = \underset{1\times1}{p}\left(E\left(\underset{T\times1}{\boldsymbol{l}}\right) - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)^{'}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(E\left(\underset{T\times1}{\boldsymbol{l}}\right) - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

$$\tilde{\boldsymbol{\psi}}_{\rho\rho} = \underset{1\times1}{p}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\eta_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)^{'}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\eta_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

Pada saat $\rho = 0$

$$\underset{1\times1}{\tilde{\boldsymbol{\psi}}_{\rho\rho}} = \underset{1\times1}{p}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\eta_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)^{'}\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\eta_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)$$

f. Elemen matriks (2,2) $\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}}$

$$\underset{(p+1)\times(p+1)}{\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}^{'}}} = -\underset{1\times1}{p}\underset{(p+1)\times T}{\boldsymbol{K}^{'}}\underset{T\times(p+1)}{\boldsymbol{K}}$$

$$\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}} = E\left(-\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}^{'}}\right)$$

$$\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}} = E\left(-\left(-\underset{1\times1}{p}\underset{(p+1)\times T}{\boldsymbol{K}^{'}}\underset{T\times(p+1)}{\boldsymbol{K}}\right)\right)$$

$$\underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}}} = \underset{1\times1}{p}\underset{(p+1)\times T}{\boldsymbol{K}^{'}}\underset{T\times(p+1)}{\boldsymbol{K}}$$

Pada saat $\rho = 0$

$$\underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}}} = \underset{1\times1}{p}\left(\underset{(p+1)\times T}{\boldsymbol{K}^{'}}\underset{T\times(p+1)}{\boldsymbol{K}}\right)$$

g. Elemen matriks (1,2) $\tilde{\boldsymbol{\psi}}_{\rho\boldsymbol{\beta}}$

$$\underset{1\times(p+1)}{\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\rho\partial\boldsymbol{\beta}^{'}}} = -2\underset{1\times1}{p}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)^{'}\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}$$

$$\tilde{\boldsymbol{\psi}}_{\rho\boldsymbol{\beta}} = E\left(-\frac{\partial^2 \mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\rho\partial\boldsymbol{\beta}^{'}}\right)$$

$$\tilde{\boldsymbol{\psi}}_{\rho\boldsymbol{\beta}} = E\left(-\left(-2\underset{1\times1}{p}\left(\underset{T\times1}{\boldsymbol{l}} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}\right)\right)$$

$$\tilde{\boldsymbol{\psi}}_{\rho\boldsymbol{\beta}} = 2\underset{1\times1}{p}\left(E\left(\underset{T\times1}{\boldsymbol{l}}\right) - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}$$

$$\underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\rho\boldsymbol{\beta}}} = 2\underset{1\times1}{p}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\eta_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{A}^{-1}}\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}$$

Pada saat $\rho = 0$

$$\underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\rho\boldsymbol{\beta}}} = 2\underset{1\times1}{p}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\eta_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}$$

h. Elemen matriks (2,1) $\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\rho}$

$$\underset{(p\times1)\times1}{\frac{\partial^2\mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}\partial\rho}} = \mathbf{0}$$

$$\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\rho} = E\left(-\frac{\partial^2\mathcal{L}(\rho,\boldsymbol{\beta},\boldsymbol{\Theta};\boldsymbol{l})}{\partial\boldsymbol{\beta}\partial\rho}\right) = \mathbf{0}$$

$$\underset{(p\times1)\times1}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\rho}} = \mathbf{0}$$

$$\underset{(p\times1)\times1}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\rho}} = \mathbf{0}$$

Invers Matriks Informasi pada saat $\rho = 0$

Jika ada matriks partisi $\mathbf{C} = \begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{C}_3 & \mathbf{C}_4 \end{pmatrix}$ maka matriks invers nya adalah $\mathbf{C}^{-1} = \begin{pmatrix} \mathbf{C}_{E1} & \mathbf{C}_{E2} \\ \mathbf{C}_{E3} & \mathbf{C}_{E4} \end{pmatrix}$ dimana

$$\mathbf{C}_{E1} = \left(\mathbf{C}_1 - \mathbf{C}_2\mathbf{C}_4^{-1}\mathbf{C}_3\right)^{-1}$$

$$\mathbf{C}_{E2} = \left(-\mathbf{C}_{E1}\mathbf{C}_2\mathbf{C}_4^{-1}\right)$$

$$\mathbf{C}_{E3} = \left(-\mathbf{C}_4^{-1}\mathbf{C}_3\mathbf{C}_{E1}\right)$$

$$\mathbf{C}_{E4} = \left(\mathbf{C}_4^{-1} - \mathbf{C}_4^{-1}\mathbf{C}_3\mathbf{C}_{E2}\right)$$

Matriks informasi $\tilde{\boldsymbol{\psi}}_\theta = \begin{pmatrix} \underset{1\times1}{\tilde{\boldsymbol{\psi}}_{\rho\rho}} & \underset{1\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\rho\boldsymbol{\beta}}} \\ \underset{(p+1)\times1}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\rho}} & \underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\beta}\boldsymbol{\beta}'}} \end{pmatrix}$

$$\tilde{\boldsymbol{\Psi}}_{\theta} = \begin{pmatrix} \underset{1\times1}{p}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\eta_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)' \underset{\underset{1\times1}{T\times T}}{\mathbf{W}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\eta_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right) & \underset{1\times(p+1)}{2\underset{1\times1}{p}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\eta_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)' \underset{T\times T}{\mathbf{W}}\underset{T\times(p+1)}{\boldsymbol{K}}} \\ \underset{(p+1)\times1}{\mathbf{0}} & \underset{(p+1)\times(p+1)}{\underset{1\times1}{p}\left(\underset{(p+1)\times T}{\boldsymbol{K}'}\underset{T\times(p+1)}{\boldsymbol{K}}\right)} \end{pmatrix}$$

$$\mathbf{C}^{-1} = \left(\begin{array}{c|c} \mathbf{C}_{E1} & \mathbf{C}_{E2} \\ \hline \mathbf{C}_{E3} & \mathbf{C}_{E4} \end{array}\right)$$

$$\mathbf{C}_{E1} = \left(\mathbf{C}_1 - \mathbf{C}_2\mathbf{C}_4^{-1}\mathbf{C}_3\right)^{-1}$$

$$\mathbf{C}_{E2} = \left(-\mathbf{C}_{E1}\mathbf{C}_2\mathbf{C}_4^{-1}\right)$$

$$\mathbf{C}_{E3} = \left(-\mathbf{C}_4^{-1}\mathbf{C}_3\mathbf{C}_{E1}\right)$$

$$\mathbf{C}_{E4} = \left(\mathbf{C}_4^{-1} - \mathbf{C}_4^{-1}\mathbf{C}_3\mathbf{C}_{E2}\right)$$

$$\underset{1\times1}{\tilde{\boldsymbol{\Psi}}_{\rho\rho}^{-1}} = \mathbf{C}_{E1} = \left(\mathbf{C}_1 - \mathbf{C}_2\mathbf{C}_4^{-1}\mathbf{C}_3\right)^{-1}$$

$$\underset{1\times1}{\tilde{\boldsymbol{\Psi}}_{\rho\rho}^{-1}} = \left(\underset{1\times1}{\tilde{\psi}_{\rho\rho}} - \underset{1\times(p+1)}{\tilde{\boldsymbol{\Psi}}_{\rho\beta}}\left(\underset{(p+1)\times(p+1)}{\tilde{\boldsymbol{\Psi}}_{\beta\beta}}\right)^{-1}\underset{(p+1)\times1}{\tilde{\boldsymbol{\Psi}}_{\beta\rho}}\right)^{-1}$$

$$\underset{1\times1}{\tilde{\boldsymbol{\Psi}}_{\rho\rho}^{-1}} = \underset{1\times1}{p^{-1}}\left[\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\eta_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)'\underset{T\times T}{\mathbf{W}}\underset{T\times T}{\mathbf{W}}\left(\underset{T\times1}{\mathbf{e}}\underset{1\times1}{\eta_t} - \underset{T\times(p+1)}{\boldsymbol{K}}\underset{(p+1)\times1}{\boldsymbol{\beta}}\right)\right]^{-1}$$

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